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GLOBAL DYNAMICS OF DISCRETE MONOTONE MAPS IN THE PLANE AND IN \mathbb{R}^N

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GLOBAL DYNAMICS OF DISCRETE MONOTONE MAPS IN THE PLANE
AND IN \mathbb{R}^N

BY
JAMES MARCOTTE

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE
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DOCTOR OF PHILOSOPHY DISSERTATION
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ABSTRACT

This dissertation investigates the local and global behavior of some monotone systems of difference equations. In each study, general results are provided as well as specific examples.

In Manuscript 2 it is shown that locally asymptotically equilibria of planar cooperative or competitive maps have basin of attraction \mathcal{B} with relatively simple geometry. The boundary of each component of \mathcal{B} consists of the union of two unordered curves, and the components of \mathcal{B} are not comparable as sets. The curves are Lipschitz if the map is of class C^1 . Further, if a periodic point is in $\partial\mathcal{B}$, then $\partial\mathcal{B}$ is tangential to the line through the point with direction given by the eigenvector associated with the smaller characteristic value of the map at the point. Examples are given.

In Manuscript 3 Sufficient conditions are given for planar cooperative maps to have the qualitative global dynamics determined solely on local stability information obtained from fixed and minimal period-two points. The results are given for a class of strongly cooperative planar maps of class C^1 on an order interval. The maps are assumed to have a finite number of strongly ordered fixed points, and also the minimal period-two points are ordered in a sense. An application is included.

In Manuscript 4 we give a characterization of monotone discrete systems of equations in terms of associated signature matrix and give some properties of certain invariant surfaces of codimension 1, which often give the boundary of attraction of some fixed points. We present several examples that illustrate our results in the case of k dimensional systems where $k \geq 3$.

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This dissertation is the result of over six years of work at the University of Rhode Island. This journey would not have been possible without the help, guidance, and support of many others. I would like to take a moment to thank them.

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Dr. Mustafa Kulenović, my co-advisor, was the first to suggest to me that I research difference equations. This led me to discover and explore a field of research I really enjoy. He has been a constant support throughout my research and I truly appreciate his guidance and wisdom.

I would also like to thank Dr. Araceli Bonifant and Dr. Richard Vaccaro for serving on my defense committee, as well as Dr. George Tsiatas, the defense chair.

Thank you to my fellow graduate students, especially Elliott Bertrand and Sarah Van Beaver. Elliott was the first person I met here at URI and was very welcoming. He was also my office-mate for the bulk of my time here and we enjoyed working together. Sarah and I took several classes together and worked on our dissertations at the same time. This journey was more meaningful and rewarding because of them.

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think about all the time. I owe all of my academic success to her as she always expected me to do my best. I wish my sister Christine all the best in her own journey through this program. Last but not least, Cassandra Czarn, who has always been there for me and supported me through this process. I'm sure I would not have finished without her support.

PREFACE

This thesis has been prepared in manuscript form. The main content of the thesis is made up of three research papers, Manuscripts 2, 3, and 4. Manuscript 2 was submitted for publication to *Discrete and Continuous Dynamical Systems, Ser.B*. Manuscript 3 was submitted for publication to *Journal of Difference Equations and Appl.* Manuscript 4 is in preparation for submission.

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MANUSCRIPT 1

Introduction

1.1 Difference Equation Basics

Difference equations describe the progression of a given quantity or population over discrete time intervals. If we consider the size of the population in the n th generation, which we denote x_n and assume that the size of the population in the $n+1$ st generation denoted x_{n+1} is a function of x_n , then we get the following *first order difference equation*

$$x_{n+1} = f(x_n) \quad n = 0, 1, \dots \quad (1)$$

where $f : \mathcal{R} \rightarrow \mathcal{R}$ is a given function. We call (1) a *one-dimensional dynamical system*. Also, the function f is called the map associated with (1). If we are given an initial value for (1), say $x_0 = d$, then applying equation (1) to x_0 multiple times results in the sequence $\{x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \dots\}$ which is called a *solution* of (1). Now the population of the $n+1$ st generation can also be dependant on the size of several previous generations $x_n, x_{n-1}, x_{n-2}, \dots$. When x_{n+1} is a function of x_n and x_{n-1} we get the following *second order difference equation*

$$x_{n+1} = f(x_n, x_{n-1}) \quad n = 0, 1, \dots \quad (2)$$

where $f : I \times I \rightarrow I$ is a given function. Similar to equation (1), we can be given initial conditions for equation (2) and find solutions.

In this thesis, we will be particularly interested in systems of difference equations which model two or more quantities or populations that depend on each other over discrete time intervals. A *two-dimensional system of difference equations* is

of the form

$$\begin{aligned} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n), \quad n = 0, 1, \dots \end{aligned} \tag{3}$$

where $f, g : \mathcal{D} \rightarrow \mathbb{R}$, $\mathcal{D} \in \mathbb{R}^2$ are given functions. Initial conditions of (3) are ordered pairs $(x_0, y_0) \in \mathcal{D}$. Systems of equations will be discussed throughout this thesis, in particular, monotone systems of equations.

In studying difference equations, the main goal is often to determine the *global dynamics* of a difference equation. Determining the global dynamics of a difference equation, or system of difference equations, is achieved by characterizing the end behavior of solutions for the equation as $n \rightarrow \infty$ for arbitrary initial conditions. The investigation of the global dynamics of a difference equation typically begins with describing the *local dynamics* of such difference equations. This analysis begins with finding *equilibrium points* of the difference equation. An equilibrium point of (3) has the form (\bar{x}, \bar{y}) and satisfies

$$\bar{x} = f(\bar{x}, \bar{y}), \quad \bar{y} = g(\bar{x}, \bar{y}).$$

For each equilibrium point of (3) we call the *basin of attraction* of (\bar{x}, \bar{y}) denoted $\mathcal{B}(\bar{x}, \bar{y})$ is defined as the set J that contains (\bar{x}, \bar{y}) such that if T is the map associated to (3) $T^n(x, y) \rightarrow (\bar{x}, \bar{y})$ as $n \rightarrow \infty$ for all $(x, y) \in J$.

There is also the potential for periodic solutions of the equation. These periodic points are often important in determining the global dynamics of the equation. A *minimal period two point* is a point $(x, y) \in \mathcal{D}$ such that $T^2(x, y) = (x, y)$ and $T(x, y) \neq (x, y)$. The same definition can be extended to points of larger period.

When conducting the local analysis of difference equations we consider the behavior of the equation about the equilibrium points, and periodic points if they exist, in a process called local stability analysis. After determining the local be-

havior, the global behavior the the equation is considered and characterized.

1.2 Local Stability Analysis

To determine the local dynamics of a difference equation we go through a process known as local stability analysis. In this process we consider each of the equilibrium points, and periodic points if they exist, of our difference equation and characterize it based on the behavior of points in a small neighborhood around the equilibrium point. We give the following definitions to characterize the different types of equilibrium points. The definitions and theorems for this section are found in [4]. For the following discussion, let $T : \mathcal{D} \rightarrow \mathcal{D}$, $\mathcal{D} \in \mathbb{R}^2$ be the map associated to (3), and let f, g be continuously differentiable functions at (\bar{x}, \bar{y}) .

Definition 1 1. An equilibrium point (\bar{x}, \bar{y}) of (3) is said to be stable if for any $\epsilon > 0$ there is $\delta > 0$ such that for every initial point (x_0, y_0) for which $\|(x_0, y_0) - (\bar{x}, \bar{y})\| < \delta$, the iterates (x_n, y_n) of (x_0, y_0) satisfy $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \epsilon$ for all $n > 0$. An equilibrium point (\bar{x}, \bar{y}) is said to be unstable if it is not stable.

2. An equilibrium point (\bar{x}, \bar{y}) of (3) is said to be locally asymptotically stable (LAS) if it is stable and if there exists $r > 0$ such that $(x_n, y_n) \rightarrow (\bar{x}, \bar{y})$ as $n \rightarrow \infty$ for all (x_0, y_0) that satisfy $\|(x_0, y_0) - (\bar{x}, \bar{y})\| < r$.

3. A periodic point (x_p, y_p) of period m is stable (respectively unstable or asymptotically stable) if (x_p, y_p) is stable (respectively unstable or asymptotically stable) fixed point of T^m .

To determine the local stability of the equilibrium points as defined above, we first find the Jacobian matrix of the map T at each (\bar{x}, \bar{y}) and use a theorem to characterize the points.

Definition 2 Let (\bar{x}, \bar{y}) be a fixed point of the map T , the Jacobian matrix of T

at (\bar{x}, \bar{y}) is the matrix

$$J_T(\bar{x}, \bar{y}) = \begin{pmatrix} \frac{\partial f}{\partial x}(\bar{x}, \bar{y}) & \frac{\partial f}{\partial y}(\bar{x}, \bar{y}) \\ \frac{\partial g}{\partial x}(\bar{x}, \bar{y}) & \frac{\partial g}{\partial y}(\bar{x}, \bar{y}) \end{pmatrix}.$$

The characteristic equation associated with the Jacobian matrix is

$$\lambda^2 - \text{tr} J_T(\bar{x}, \bar{y})\lambda + \det J_T(\bar{x}, \bar{y}) = 0$$

The following two theorems will provide criteria to easily characterize the equilibrium points of (3).

Theorem 1 *Let $T = (f, g)$ be a continuously differentiable function defined on an open set W in \mathbb{R}^2 , and let (\bar{x}, \bar{y}) in W be a fixed point of T .*

1. *If all the eigenvalues of the Jacobian matrix $J_T(\bar{x}, \bar{y})$ have modulus less than one, then the equilibrium point (\bar{x}, \bar{y}) is locally asymptotically stable.*
2. *If at least one of the eigenvalues of the Jacobian matrix $J_T(\bar{x}, \bar{y})$ has modulus greater than one, then the equilibrium point (\bar{x}, \bar{y}) is unstable.*

Definition 3 1. *If both eigenvalues of the Jacobian matrix $J_T(\bar{x}, \bar{y})$ have modulus bigger than one, such a fixed point is called a source or repeller.*

2. *If one eigenvalue of the Jacobian matrix $J_T(\bar{x}, \bar{y})$ has modulus less than one and the other eigenvalue has modulus bigger than one, such a fixed point is called a saddle.*

Theorem 2 1. *An equilibrium point (\bar{x}, \bar{y}) of (3) is locally asymptotically stable if every solution of the characteristic equation lies inside the unit circle, that is, if*

$$|\text{tr} J_T(\bar{x}, \bar{y})| < 1 + \det J_T(\bar{x}, \bar{y}) < 2.$$

2. Similarly, an equilibrium point (\bar{x}, \bar{y}) of (3) is locally a repeller if every solution of the characteristic equation lies outside the unit circle, that is, if

$$|trJ_T(\bar{x}, \bar{y})| < |1 + detJ_T(\bar{x}, \bar{y})| \text{ and } |detJ_T(\bar{x}, \bar{y})| > 1.$$

3. An equilibrium point (\bar{x}, \bar{y}) of (3) is locally a saddle point if the characteristic equation has one root that lies inside the unit circle and one root that lies outside the unit circle, that is, if

$$|trJ_T(\bar{x}, \bar{y})| > |1 + detJ_T(\bar{x}, \bar{y})|$$

By using these theorems we can characterize each of the equilibrium points of (3) which gives the local dynamics of (3).

1.3 Monotone Systems of Difference Equations

Since this thesis will be discussing monotone systems of difference equations throughout, we also provide basic definitions and theorems for monotone systems. Consider (3) with T the map associated with (3). Then we have the following definitions and theorems about monotone systems.

Definition 4 The North-east partial order \leq_{NE} is defined by $(x, y) \leq_{NE} (w, z)$ if and only if $x \leq w$ and $y \leq z$. Also set $(x, y) <_{NE} (w, z)$ if $(x, y) \leq_{NE} (w, z)$ and $(x, y) \neq (w, z)$, and $(x, y) \ll_{NE} (w, z)$ if and only if $x < w$ and $y < z$. The South-east partial order \leq_{SE} is defined by $(x, y) \leq_{SE} (w, z)$ if and only if $x \leq w$ and $y \geq z$. The symbols $<_{SE}$ and \ll_{SE} are similarly defined to $<_{NE}$ and \ll_{NE} .

Definition 5 A map T is called cooperative if $T(x, y) \leq_{NE} T(w, z)$ whenever $(x, y) \leq_{NE} (w, z)$. T is called strongly cooperative if $T(x, y) \ll_{NE} T(w, z)$ whenever $(x, y) <_{NE} (w, z)$. Similarly, a map T is called competitive if $T(x, y) \leq_{SE}$

$T(w, z)$ whenever $(x, y) \leq_{SE} (w, z)$. T is called *strongly competitive* if $T(x, y) <_{SE} T(w, z)$ whenever $(x, y) <_{SE} (w, z)$.

Definition 6 *The order interval $[p, q]_{NE}$ is the set $[p, q]_{NE} = \{x \in \mathbb{R}^2 : p \leq_{NE} x \leq_{NE} q\}$. This definition is similar for a South-east order interval.*

The following theorem and corollary is a theorem of Dancer and Hess in [2].

Theorem 3 *(The Order Trichotomy Theorem) Let $X = [a, b]$, where $a < b$. Let the map $T : X \rightarrow X$ be monotone and $T(X)$ have compact closure in X . Then, at least one of the following holds:*

1. *There is a fixed point c such that $a < c < b$.*
2. *There exists an entire orbit from a to b that is increasing, and strictly increasing if T is strictly monotone.*
3. *There exists an entire orbit from b to a that is decreasing, and strictly decreasing if T is strictly monotone.*

Corollary 1 *Let $X = [a, b]$, where $a < b$ and a, b are stable fixed points. Let the map $T : X \rightarrow X$ be monotone and $T(X)$ have compact closure in X . Then there is a third fixed point in $[a, b]$.*

These definitions and theorem are essential for understanding the analysis of monotone systems. The focus of this thesis is to provide some general results for the basins of attraction of fixed points in some planar cooperative maps.

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MANUSCRIPT 2

**Properties of Basins of Attraction for Planar Discrete Cooperative
Maps**

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Abstract

It is shown that locally asymptotically equilibria of planar cooperative or competitive maps have basin of attraction \mathcal{B} with relatively simple geometry: the boundary of each component of \mathcal{B} consists of the union of two unordered curves, and the components of \mathcal{B} are not comparable as sets. The curves are Lipschitz if the map is of class C^1 . Further, if a periodic point is in $\partial\mathcal{B}$, then $\partial\mathcal{B}$ is tangential to the line through the point with direction given by the eigenvector associated with the smaller characteristic value of the map at the point. Examples are given.

2.1 Introduction and Preliminaries

Fixed points and periodic points of planar maps often have basins of attraction that have very complex boundary. This is the case even if the map is smooth. An example is the planar map $F(x, y) = (x^2 - y^2 - 1, 2xy)$, $(x, y) \in \mathbb{R}^2$ (on the complex plane \mathbb{C} , $f(z) = z^2 - 1$) which has two repelling fixed points and a single minimal period-two pair, namely $\{(-1, 0), (0, 0)\}$. The basin of attraction of the minimal period-two pair has fractal boundary [13, 14]. See [12, 14, 15, 16, 17] for further properties of the basins of attraction for general maps in the plane or in higher dimension.

In this paper we consider maps $T(x, y) = (f(x, y), g(x, y))$, where f and g are continuous functions defined on some subset of \mathbb{R}^2 with non-empty interior, such that f and g are non-decreasing in all of its arguments. Such maps are said to be *cooperative*. It is shown in this paper that, in stark contrast to the general case of planar maps, basins of attraction \mathcal{B} of fixed points and periodic points of cooperative maps have simple geometry. In particular, when \mathcal{B} contains a neighborhood of the periodic orbit, it is then bounded by unordered curves (in the sense of north-east order), which is to say that they are the graphs of decreasing functions. Moreover, at any fixed or periodic points on $\partial\mathcal{B}$, the latter is tangential

to a line with direction of an eigenvector associated with a characteristic value of the map at the point in question. If \mathcal{B} has more than one connected component, then any two components are non-comparable, and if the map is of class C^1 , then the curves bounding \mathcal{B} are Lipschitz.

As a motivating example consider the difference equation from [1]

$$x_{n+1} = x_n^3 + x_{n-1}^3 \quad x_{-1}, x_0 \in \mathbb{R}, \quad n = 0, 1, \dots, \quad (4)$$

which has associated map

$$F(x, y) = (y, x^3 + y^3) \quad (x, y) \in \mathbb{R}^2. \quad (5)$$

The fixed points of the map are $(0, 0)$, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, where the origin is locally asymptotically stable other two fixed points are saddle points. There are no periodic points. By using results from [9], it is shown in [1] that the basin of attraction \mathcal{B} of $(0, 0)$ is unbounded, and it consists of the union of the stable manifolds of the two nonzero fixed points, see Fig. 2.1(a). Notice that F is cooperative and F^2 is strongly cooperative.

A variation on (4) is the difference equation

$$x_{n+1} = x_n^3 + x_{n-1}^9, \quad x_{-1}, x_0 \in \mathbb{R}, \quad n = 1, 2, \dots \quad (6)$$

whose associated map

$$G(x, y) = (y^3, x^3 + y^3), \quad (x, y) \in \mathbb{R}^2, \quad (7)$$

has three fixed points: the point $(0, 0)$ which is LAS, and the points

$(-0.617, -0.851)$ and $(0.617, 0.851)$ which are saddle points. In addition, there are two repelling minimal period-two points $(-1.349, 1.105)$, $(1.349, -1.105)$. It can be shown with results from [9] that the basin \mathcal{B} of the origin is bounded, and that $\partial\mathcal{B}$ consists of the union of stable manifolds of the two nonzero fixed points, and that the period two points are endpoints to both manifolds. See figure Figure 2.1 (b). The map G is cooperative and G^2 is strongly cooperative.

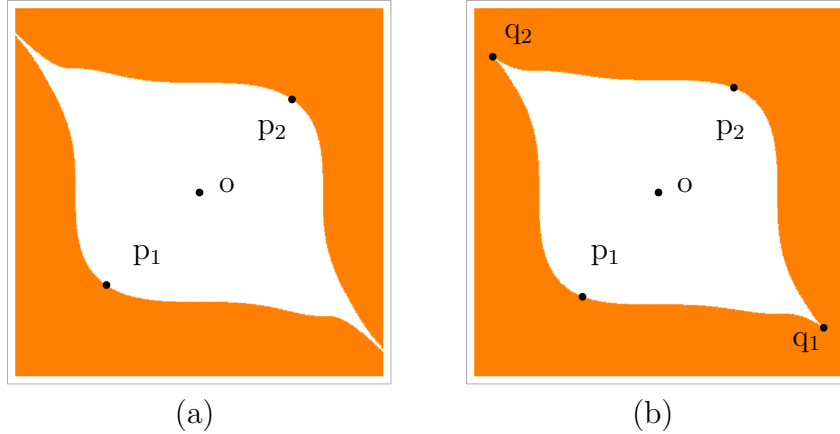


Figure 1. (a) The basin of attraction \mathcal{B} of the zero fixed point o of the map $T(x, y) = (y, x^3 + y^3)$. Note that \mathcal{B} is unbounded, and $\partial\mathcal{B}$ contains two fixed points p_1 and p_2 which are saddle points. The union of the stable manifolds of p_1 and p_2 gives $\partial\mathcal{B}$. (b) The basin of attraction \mathcal{B} of the zero fixed point of the map $T(x, y) = (y^3, x^3 + y^3)$. The set \mathcal{B} is bounded, and $\partial\mathcal{B}$ contains two fixed points p_1 and p_2 (saddles) and a repelling minimal period-two pair q_1 and q_2 . The union of the stable manifolds of p_1 and p_2 gives $\partial\mathcal{B}$.

The previous examples suggest the question of whether the geometry of the basin of locally asymptotically stable fixed or periodic points of planar monotone maps is particularly simple and amenable to a “nice” characterization.

We note that the maps in (5) and (7) are (locally) invertible, and that in each of both cases the boundary of the basin \mathcal{B} of the origin contains two saddle points. This allows, by using the results from [9] for example, the characterization of $\partial\mathcal{B}$ as the union of stable manifolds of the saddle points. However, local invertibility of a cooperative or competitive map is not always true. Also, there is the question of the components of the basin of attraction in other cases, in addition to the

possible presence of other fixed points (perhaps nonhyperbolic) on the boundary of the basin.

In general, the basin of attraction $\mathcal{B}(E)$ of locally asymptotically stable fixed point E of a map T satisfies

$$\mathcal{B}(E) = \bigcup_{k=0}^{\infty} T^{-k}\mathcal{B}_0(E), \quad (8)$$

where $\mathcal{B}_0(E)$ is a largest connected invariant set containing E , and T^0 is the identity function. The problem of characterization of $\mathcal{B}(E)$ is finding the properties of $T^{-k}\mathcal{B}_0(E)$ for an arbitrary map. In this paper we show that if T is a monotone (cooperative or competitive) map, one can characterize those components of the basin of attraction. Our main results will show that the previous two examples are indicative of the structure of such basin of attraction. In addition, the components of the basin form an unordered chain of non-invariant sets which eventually map into $\mathcal{B}_0(E)$. These components will be ordered in the south-east ordering which we define next.

This paper is organized as follows. In the rest of this section we give some basic notions about monotone maps in the plane. The second section presents our main results and some corollaries. The third section presents examples and the fourth section gives proofs of the main results.

Consider a partial ordering \preceq on \mathbb{R}^2 . Two points $x, y \in \mathbb{R}^2$ are said to be related if $x \preceq y$ or $y \preceq x$. Also, a strict inequality between points may be defined as $x \prec y$ if $x \preceq y$ and $y \neq x$. A stronger inequality may be defined as $x \ll y$ if $x \preceq y$ with $x_1 \neq y_1$ and $x_2 \neq y_2$. If $x \preceq y$, the order interval $[x, y]$ is the set $\{z : x \preceq z \preceq y\}$. A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ is a continuous function $T : \mathcal{R} \rightarrow \mathcal{R}$. A point \bar{x} in \mathcal{R} is a *fixed point* of T if $T(\bar{x}) = \bar{x}$. The *basin of attraction* of a fixed point \bar{x} of a map T , denoted as $\mathcal{B}(\bar{x})$, is defined

as the set of all initial points x_0 for which the sequence of iterates $T^n(x_0)$ converges to \bar{x} . The map T is monotone if $x \preceq y$ implies $T(x) \preceq T(y)$ for all $x, y \in \mathcal{R}$, and it is strongly monotone on \mathcal{R} if $x \prec y$ implies that $T(x) \ll T(y)$ for all $x, y \in \mathcal{R}$. The map is strictly monotone on \mathcal{R} if $x \prec y$ implies that $T(x) \prec T(y)$ for all $x, y \in \mathcal{R}$. Throughout this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the *South-East* (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$. A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is called *cooperative* and a map monotone with respect to the South-East ordering is called *competitive*. If T is continuously differentiable on an open set, a sufficient condition for T to be strongly cooperative (respectively, strongly competitive) is that at every point of the set, the jacobian matrix has positive entries (resp. positive diagonal entries and negative off-diagonal entries). For $x \in \mathbb{R}^2$, define $Q_i(x)$ for $i = 1, \dots, 4$ to be the usual four quadrants based at x and numbered in a counterclockwise direction, for example, $Q_1(x) = \{y \in \mathbb{R}^2 : x \preceq_{ne} y\}$. A set A is said to be *order convex* if for every $x, y \in A$, the order interval $[x, y]$ is a subset of A . A general reference for difference equations and maps is [2]. For some basic notions about monotone discrete systems in the plane, see [1, 5, 6, 7, 8, 9, 10, 18].

2.2 Main Results

The main result applies to cooperative maps on an order interval whose k -th power (for some $k \geq 1$) is strongly cooperative. Smoothness of the map is not assumed, but it is considered later in Theorems 5 and 6. Unbounded domains are discussed in Remark 2, competitive maps in Remark 3, and periodic points in Remark 5.

Theorem 4 *Let \mathcal{R} be an order interval in \mathbb{R}^2 with nonempty interior, and let*

$T : \text{int}(\mathcal{R}) \rightarrow \text{int}(\mathcal{R})$ be a cooperative map whose k -th power (for some $k \geq 1$) is strongly cooperative. Suppose $\bar{x} \in \mathcal{R}$, and set $\mathcal{B} := \{x \in \text{int}(\mathcal{R}) : T^m(x) \rightarrow \bar{x} \text{ as } m \rightarrow \infty\}$. If there exists an open set \mathcal{O}' in \mathbb{R}^2 containing \bar{x} such that $\mathcal{O} := \mathcal{O}' \cap \text{int}(\mathcal{R}) \subset \mathcal{B}$, then

- (i) The boundary of each connected component \mathcal{B}' of \mathcal{B} is the union of two curves \mathcal{C}_- and \mathcal{C}_+ (termed the lower and upper boundary curves of \mathcal{B}' , respectively). Points on a boundary curve that are interior to \mathcal{R} are non-comparable. The boundary curves \mathcal{C}_- and \mathcal{C}_+ have common endpoints, and these are their only common points.
- (ii) If \mathcal{B}' and \mathcal{B}'' are any two distinct components of \mathcal{B} , then either $\mathcal{B}' \ll_{se} \mathcal{B}''$ or $\mathcal{B}'' \ll_{se} \mathcal{B}'$.
- (iii) Denote with \mathcal{B}_* the connected component of \mathcal{B} whose closure in \mathcal{R} contains \bar{x} . The set \mathcal{B}_* is T -invariant. The intersection of each boundary curve of \mathcal{B}_* with the interior of \mathcal{R} is T -invariant.
- (iv) If \mathcal{B}' is a component of \mathcal{B} such that $\mathcal{B}' \neq \mathcal{B}_*$, then there exists a positive integer n that depends on \mathcal{B}' such that $T^{n-1}(\mathcal{B}') \cap \mathcal{B}_* = \emptyset$ and $T^n(\mathcal{B}') \subset \mathcal{B}_*$. If \mathcal{C}'_{\pm} and \mathcal{C}_{\pm} are the boundary curves of \mathcal{B}' and \mathcal{B}_* respectively, set $\tilde{\mathcal{C}}'_{\pm} := \mathcal{C}'_{\pm} \cap \text{int}(\mathcal{R})$ and $\tilde{\mathcal{C}}_{\pm} := \mathcal{C}_{\pm} \cap \text{int}(\mathcal{R})$. Then $T^n(\tilde{\mathcal{C}}'_-) \subset \tilde{\mathcal{C}}_-$ and $T^n(\tilde{\mathcal{C}}'_+) \subset \tilde{\mathcal{C}}_+$.

Remark 1 If in Theorem 4 the point \bar{x} is in $\text{int}(\mathcal{R})$, then \bar{x} is a fixed point and the set \mathcal{B} is the *basin of attraction* of \bar{x} in $\text{int}(\mathcal{R})$. However, the map need not be defined at \bar{x} for Theorem 4 to apply, see Example 2 in Section 2.3.

Remark 2 The conclusions of Theorem 4 are valid for maps T on *unbounded* domains \mathcal{R} of either one is of the forms $\{x : x \preceq p\}$, $\{x : p \preceq x\}$, or \mathbb{R}^2 . To prove this, consider the natural extension of the partial order to the extended plane

$\bar{\mathbb{R}}^2 = [-\infty, \infty] \times [-\infty, \infty]$. The set \mathcal{R} is a subset of $\bar{\mathbb{R}}^2$. Also modify the notion of boundary curve so that points common to a boundary curve and the boundary of the domain of the map may have one or both coordinates equal to $-\infty$ or $+\infty$. The proof is essentially the same as that for Theorem 4. See Example 1 in Section 2.3, where the domain of the map is \mathbb{R}^2 .

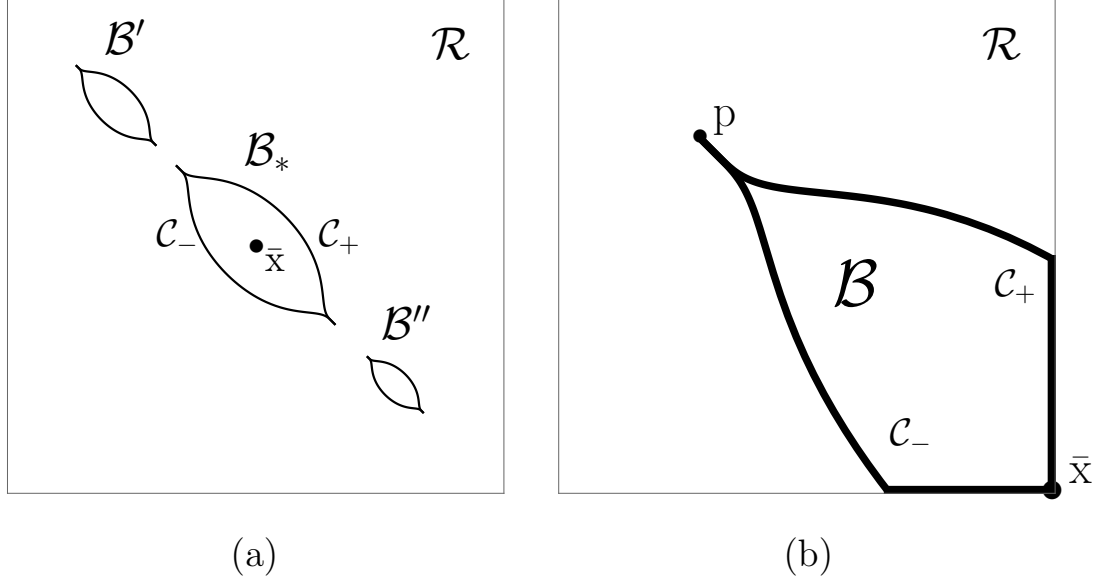


Figure 2. (a) the basin \mathcal{B} of the fixed point \bar{x} has three components \mathcal{B}' , \mathcal{B}_* , \mathcal{B}'' whose closure is in $\text{int}(\mathcal{R})$ and such that $\mathcal{B}' \ll_{se} \mathcal{B}_* \ll_{se} \mathcal{B}''$. Each component has boundary curves \mathcal{C}_+ and \mathcal{C}_- . (b) The set \mathcal{B} has only one component, which has part of its boundary in $\partial\mathcal{R}$. Also, $\bar{x} \in \partial\mathcal{R}$. The point p is an endpoint of both boundary curves \mathcal{C}_- and \mathcal{C}_+ . The point p is a fixed point of T .

By (i) and (iii) of Theorem 4, the set of endpoints of the boundary curves \mathcal{C}_+ and \mathcal{C}_- of \mathcal{B}_* that belong to $\text{int}(\mathcal{R})$ is invariant. Such set has at most two points in $\text{int}(\mathcal{R})$, hence any such point is periodic with period two. Therefore we have the following result.

Corollary 2 *Let p be an endpoint of a boundary curve of \mathcal{B}_* . If p is in $\text{int}(\mathcal{R})$, then p is a fixed point or a minimal period-two point of T .*

From (ii.) of Theorem 4 and Corollary 2 we have the following result.

Corollary 3 *If there are no period-two points in $\mathcal{Q}_2(\bar{x}) \cup \mathcal{Q}_4(\bar{x})$ other than \bar{x} , then there is only one component of \mathcal{B} , and the corresponding boundary curves have endpoints on \mathcal{R} .*

Smoothness of the map implies that the boundary of the basin \mathcal{B} in Theorem 4 is guaranteed to have additional properties.

Theorem 5 *Let \mathcal{R} , T and \mathcal{B} be as in Theorem 4. Assume the hypotheses of Theorem 4. Suppose z is a minimal period k point of T in $\text{int}(\mathcal{R}) \cap \partial\mathcal{B}$, and that T is of class C^1 in a neighborhood of z . If the jacobian matrix of T^k at z has positive entries, then $\partial\mathcal{B}$ is tangential at z to the line ℓ with direction given by the eigenspace associated to the characteristic value of T at z with the smallest modulus.*

Theorem 6 *Assume the hypotheses of Theorem 4. Suppose T is a continuously differentiable map on $\text{int}(\mathcal{R})$ such that the jacobian matrix at every point in $\text{int}(\mathcal{R})$ has positive entries. Let \mathcal{B}' be a component of the basin \mathcal{B} of \bar{x} , and let \mathcal{C}_- and \mathcal{C}_+ be the corresponding boundary curves. Then,*

- i. *Each of the curves \mathcal{C}_- and \mathcal{C}_+ of \mathcal{B}' is the graph of a Lipschitz function of a real variable.*
- ii. *If \mathcal{C}_- and \mathcal{C}_+ intersect at a hyperbolic periodic point $p \in \text{int}(\mathcal{R})$, then p is a source.*

Remark 3 A version of Theorems 4, 5 and 6 and corollaries 2 and 3 are valid for maps T that are *competitive* (instead of cooperative). To obtain these results, replace the word *cooperative* by the word *competitive*, and replace the *north-east partial order* by the *south-east partial order* and vice-versa. With these modifications, the proofs carry over word for word, so those will be omitted. See Example 2 in Section 2.3.

Remark 4 If the boundary of the set \mathcal{B}_* in Theorem 4 has a fixed or periodic saddle point, the local stable manifold can be extended to a global stable manifold by using topological arguments or results such as those in [9]. In these cases it is possible to obtain a description of $\partial\mathcal{B}_*$. But often the sufficient conditions for global stable manifold are difficult to verify or are not applicable at all. In these cases, Theorems 4, 5, 6 and corollaries give the existence of invariant Lipschitz curves where other methods fail.

Remark 5 The results of this section are applicable to locally asymptotically stable minimal period k points p of a map T . To do this, consider the iterates $p, T(p), \dots, T^{k-1}(p)$ as a fixed points of T^k . The basin of the orbit of p is then the union of the basins of points of the orbit as fixed points of T^k .

2.3 Examples

In this section we provide two applications. Example 1 is a discussion on the global dynamics of a strongly cooperative map whose domain is \mathbb{R}^2 . We show that the origin is LAS, with basin of attraction that has more than one component. Admittedly the example is somewhat contrived, but it is the only example of cooperative map known to the authors with the property that the basin of attraction of a point consists of several components. A feature of the method used to produce the example is that it can be used to generate other examples with basins of attraction consisting of many components, even a countably infinite number of them. In Example 2 we consider a class of parametrized competitive maps defined on the nonnegative quadrant minus the origin. The maps have the origin as a singular point that has a substantial set attracted to it. Our results in this paper can be applied to characterize the boundary of the set attracted to the origin. This characterization is valid for all values of the parameters.

Example 1 We begin by defining a cooperative map U on the plane for which the origin is a LAS fixed point with unbounded basin of attraction. Then a map V is defined as a specific perturbation of U , so that the origin has bounded basin of attraction consisting of three components.

Consider the map

$$U(x, y) := (0.5(x + y) + x^3 + y^3, 0.35(x + y) + x^5 + y^5), \quad (x, y) \in \mathbb{R}^2. \quad (9)$$

This is a strongly cooperative map for which the origin is LAS, as can be easily determined from analysis of the jacobian matrix. The basin \mathcal{B} of the origin consists of a single unbounded component. This is a consequence of the relation $T(x, -x) = (0, 0)$ for $x \in \mathbb{R}$, hence the line $\{(x, -x) : x \in \mathbb{R}\}$ is a subset of \mathcal{B} . That there cannot be any other components now follows from Theorem 4. See Figure 3.

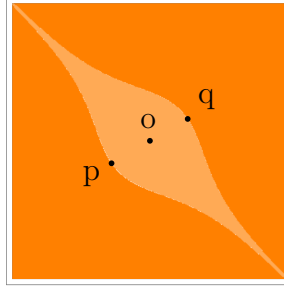


Figure 3. Basin of attraction of the origin o for the map U in (9). The points p and q are saddle fixed points.

We now consider a perturbation of U of the form

$$V(x, y) = U(x, y) + \Delta(x, y). \quad (10)$$

We shall choose Δ so that V is a strongly cooperative map with the origin being a LAS fixed point with basin of attraction having more than one component. One

way to accomplish this is by further specializing Δ to have the form

$$\Delta(x, y) := \left(\frac{\phi(x) - \phi(y)}{2}, \frac{-\phi(x) + \phi(y)}{2} \right) = \frac{1}{2} (\phi(x) - \phi(y)) (1, -1), \quad (11)$$

where ϕ is a smooth real valued odd function of a real variable to be chosen later. Since ϕ is an odd function we have,

$$\Delta(x, -x) = (\phi(x), -\phi(x)) = \phi(x) (1, -1). \quad (12)$$

Since $U(x, -x) = (0, 0)$, the dynamics of $V(x, y)$ on the line $x + y = 0$ are exactly the dynamics of ϕ on the real line.

We shall require that $\phi(0) = 0$, which is necessary for the origin to be a fixed point of $V(x, y)$. Also desirable is a small value of $|\phi'(0)|$ so the origin retains local stability after perturbing the original map. The function ϕ must give a cooperative V , which can be ensured by choosing ϕ with suitable growth restrictions. Consider the function (see Figure 4)

$$\phi(t) := \frac{0.00075 t^7 + 2.5 t^3}{(0.1 t^2 + 1)^2 (t^2 + 1)} \quad (13)$$

With ϕ as in (13) the map $V(x, y)$ is strongly cooperative on its domain. See Figure 5 for a graphical illustration. The map V has a locally asymptotically stable fixed point $o(0, 0)$ and saddle fixed points $r(-0.404, -0.297)$ and $s(0.404, 0.297)$ as well as the period-two points $q_1(-0.953, 0.953)$, $p_1(0.953, -0.953)$, $q_2(-2.067, 2.067)$, $p_2(2.067, -2.067)$ and eventually period-two points $p_3(6.034, -6.034)$, $q_3(-6.034, 6.034)$, $q_4(-12.798, 12.798)$, $p_4(12.798, -12.798)$. See Figure 6. The invariant component of the basin of

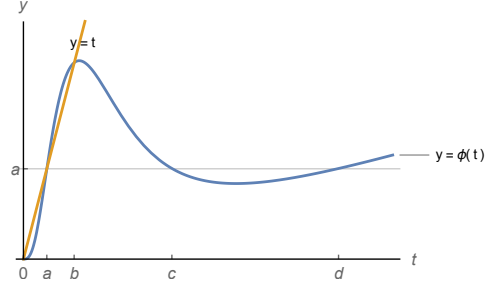


Figure 4. Graphs of ϕ from (13) and the identity function on the nonnegative semi axis. ϕ has locally asymptotically stable fixed points 0, $b = 2.06$, and a repelling fixed point $a = 0.95$. The real numbers $c = 6.03$ and $d = 12.80$ are pre-images of a . The basin of attraction of 0 on the semi-axis consists of the intervals $0 \leq t < a$ and $c < t < d$. All decimal numbers have been rounded to two decimals.

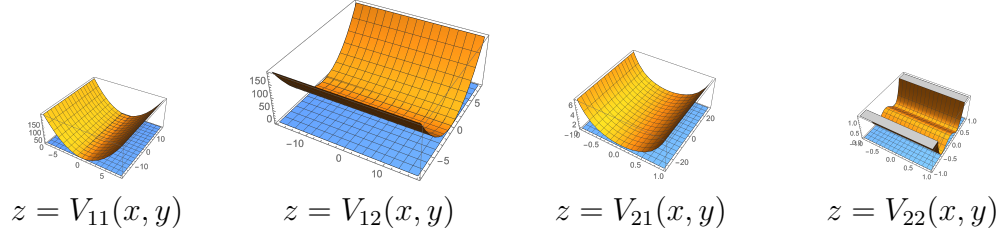


Figure 5. The partial derivatives of $V(x, y)$. Also shown in the plane $z = 0$.

attraction of the origin is bounded by the global stable manifolds of two saddle fixed points which have endpoints at period-two points.

Example 2 Consider maps of the form

$$T(x, y) := \left(\frac{x^3}{\alpha x + (1 - \alpha)y}, \frac{y^3}{(1 - \delta)x + \delta y} \right), \quad (x, y) \in \mathbb{R}_+^2 \setminus \{0, 0\}, \quad \alpha, \delta \in (0, 1) \quad (14)$$

The map T is competitive on its domain and strongly competitive on its interior, the open positive quadrant, as can be concluded from the jacobian matrix

$$\begin{pmatrix} \frac{x^2 (2x\alpha + 3y(1-\alpha))}{((\alpha x + (1-\alpha)y)^2)} & -\frac{x^3 (1-\alpha)}{((\alpha x + (1-\alpha)y)^2)} \\ -\frac{y^3 (1-\delta)}{((\delta x + \delta y)^2)} & \frac{y^2 (2y\delta - 3x(\delta-1))}{((1-\delta)x + \delta y)^2} \end{pmatrix} \quad (15)$$

The origin o is a singular point, and there are three fixed points, namely $a(\alpha, 0)$, $d(0, \delta)$ and $b(1, 1)$. A straightforward calculation gives that a and d are

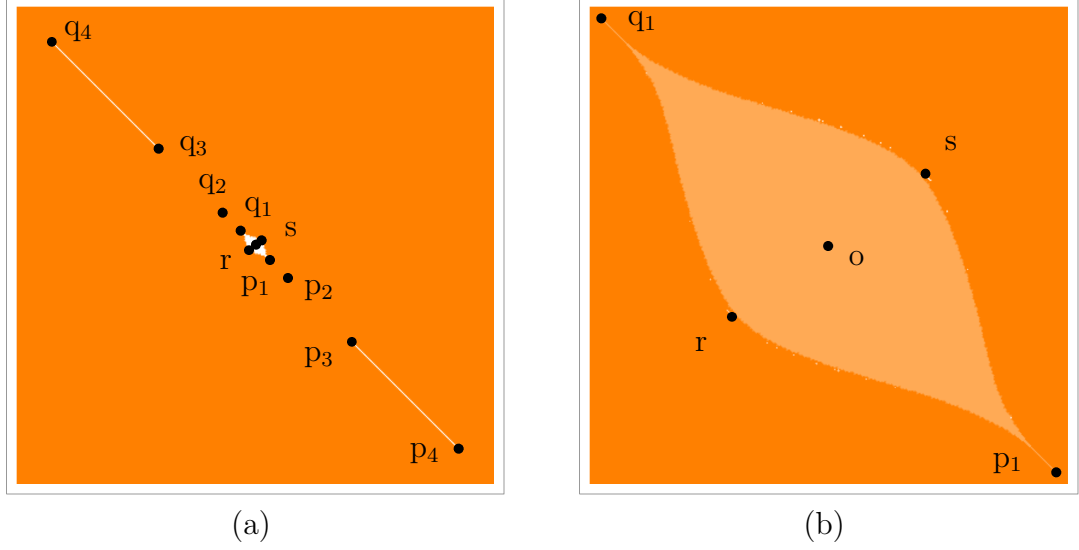


Figure 6. (a) Three components of the basin of attraction of the fixed point zero of the map $V(x, y)$ in Example 1. Here r, s are saddle fixed points, p_1 and q_1 are a saddle period-two pair, p_2 and q_2 are repelling fixed points, and p_3, p_4, q_3, q_4 are eventual period-two points. The boundary of the invariant part of the basin of attraction consist of stable manifolds of saddle fixed points with a period-two endpoints. In addition, there are two eventually period-two points which are end points of another piece of the basin of attraction which is mapped into the invariant part. (b) The invariant component of the basin of the origin o .

saddle points, each with an open semiaxis as unstable manifold. Also b is a repeller, with characteristic values $2, 4 - \alpha - \delta$, and corresponding eigenvectors $(1, 1)$ and $(\alpha - 1, 1 - \delta)$. The ray $\{(x, x) : x > 0\}$, is invariant, more specifically we have

$$T(x, x) = (x^2, x^2) \quad \text{for all } x > 0, \quad \alpha, \delta \in (0, 1). \quad (16)$$

The following is a complete characterization of the global dynamics of map (14) for all allowed values of the parameters. See Figure 7.

Proposition 1 *Let T be as in (14). For all values of α and δ in $(0, 1)$, the set $\mathcal{B} := \{(x, y) : T^n(x, y) \rightarrow (0, 0)\}$ is bounded by north-east ordered Lipschitz curves \mathcal{C}_+ and \mathcal{C}_- , which have endpoints a, b and d, b respectively. Also, \mathcal{C}_+ and \mathcal{C}_- are tangential to the line $y = x$ at the point b . If $(x, y) \neq b$ is in \mathcal{C}_+ (resp. \mathcal{C}_-) then*

$T^n(x, y) \rightarrow a$ (resp. $T^n(x, y) \rightarrow d$), while if (x, y) is in the complement of the closure of \mathcal{B} , then $\|T^n(x, y)\| \rightarrow \infty$.

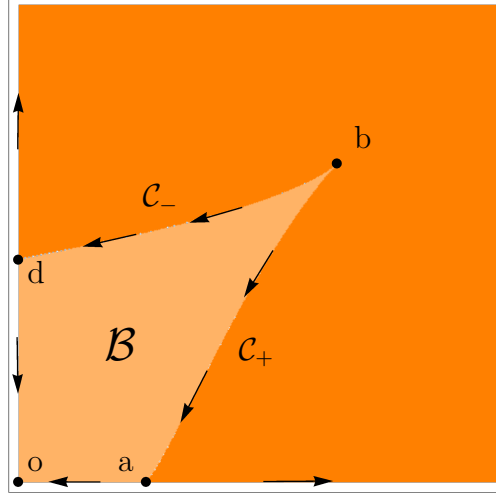


Figure 7. Global dynamics for map (14) as given in Proposition 1. Here $\alpha = 0.4$ and $\delta = 0.7$.

Proof. We begin by verifying that the origin has a relative neighborhood that is a subset of \mathcal{B} . This can be seen as follows. The relations $T(x, 0) \preceq_{se} (x, 0)$ for $0 < x < \alpha$, and $(0, y) \preceq_{se} (0, y)$ for $0 < y < \delta$ imply that for (u, v) with $0 < u < x$ and $0 < v < y$, $T^n(0, y) \preceq_{se} T^n(u, v) \preceq_{se} T^n(x, 0)$. Since $T^n(x, 0) \rightarrow (0, 0)$ and $T^n(0, y) \rightarrow (0, 0)$, we have $T^n(u, v) \rightarrow (0, 0)$. Thus the set $\mathcal{O}' = \{(x, y) : 0 < x < \alpha, 0 < y < \delta\}$ satisfies $\mathcal{O}' \subset \mathcal{B}$. Therefore the hypotheses of Theorems 4, 5 and 6 are satisfied.

We now show that \mathcal{B} has only one component. By Theorem 4, all components of \mathcal{B} are non-comparable in the south-east ordering, therefore they are comparable in the north-east ordering. By (16) the open line segment $\mathcal{L} := \{(x, x) : 0 < x < 1\}$ consists of points (x, x) such that $T^n(x, x) \rightarrow (0, 0)$. Also by (16) the ray $\mathcal{S} := \{(x, x) : 1 < x < \infty\}$ consists of points (x, x) such that $\|T^n(x, x)\| \rightarrow \infty$. For any point (z, w) with $z > 1$ or $w > 1$ one may choose x so that $(x, x) \preceq_{se} (z, w)$ or

$(z, w) \preceq_{se} (x, x)$. It follows $T^n(x, x) \preceq_{se} T^n(z, w)$ or $T^n(z, w) \preceq_{se} T^n(x, x)$. Since $T^n(x, x) = (x^{2^n}, x^{2^n})$, we have $\|T^n(z, w)\| \rightarrow \infty$. In particular, it follows that \mathcal{B} has only one component.

Note $\{a, d, b\} \subset \partial\mathcal{B}$. Let \mathcal{C}_- and \mathcal{C}_+ be as in Theorem 4. Since no points outside of the unit square belong to \mathcal{B} , it follows that b is an endpoint of both \mathcal{C}_+ and \mathcal{C}_- . Also a is an endpoint of \mathcal{C}_- and d is an endpoint of \mathcal{C}_+ , due to the fact that the axes are unstable manifolds of a and d . The rest of the proposition follows from Theorems 4, 5 and 6, and their corollaries. \square

2.4 Proofs

Proof of Theorem 4. It is sufficient to consider the case where T is strongly monotonic. To see this, let T , \mathcal{B} , k and \mathcal{O} be as in Theorem 4, and let $\mathcal{B}_k := \{x \in \text{int}(\mathcal{R}) : T^{mk}(x) \rightarrow \bar{x} \text{ as } m \rightarrow \infty\}$. If $x \in \mathcal{B}_k$, then $T^{mk}(x) \in \mathcal{O}$ for m large enough, which implies $x \in \mathcal{B}$. Thus $\mathcal{B}_k \subset \mathcal{B}$, and since $\mathcal{B} \subset \mathcal{B}_k$ it follows $\mathcal{B} = \mathcal{B}_k$. Without loss of generality we assume for the rest of this section that T is a strongly monotonic map ($k = 1$).

We prove several claims first. The first two claims are about certain properties of \mathcal{B} and its boundary set.

Claim 1 *The set \mathcal{B} is open and order convex, and it has either a finite or countably infinite number of connected components.*

Proof. If $x \in \mathcal{B}$, then for sufficiently large $m \in \mathbb{N}$ we have $T^m(x) \in \mathcal{O}$. Then x is an element of $(T^m)^{-1}(\mathcal{O})$, which is an open subset of \mathcal{B} . Thus \mathcal{B} is open. If $\{x, z\} \subset \mathcal{B}$, then by monotonicity of T , for every $y \in \text{int}(\mathcal{R})$ and all $m \in \mathbb{N}$, $x \preceq y \preceq z$ implies $T^m(x) \preceq T^m(y) \preceq T^m(z)$. Hence $T^m(y) \rightarrow \bar{x}$ and we conclude \mathcal{B} is order-convex. If the number of connected components of \mathcal{B} is not finite, choose a point in each of the components with rational entries. The collection of such points is countable, hence so is the collection of components of \mathcal{B} . \square

Claim 2 *The set $\partial\mathcal{B}$ does not contain a linearly ordered line segment contained in $\text{int}(\mathcal{R})$.*

Proof. Arguing by contradiction, suppose $\partial\mathcal{B}$ contains a \preceq_{ne} linearly ordered line segment $L(x, z) \subset \text{int}(\mathcal{R})$. Choose y a point in $L(x, z)$ with $y \neq x, z$. Then $T(x) <_{ne} T(y) \ll_{ne} T(z)$ by strong monotonicity of T . But then $V \ll_{ne} T(y) \ll_{ne} W$ for some open neighborhoods V of $T(x)$ and W of $T(z)$. Now both V and W contain points in \mathcal{B} , say v and w . In particular, $v \ll_{ne} T(y) \ll_{ne} w$. Since \mathcal{B} is order-convex, it follows that $T(y) \in \mathcal{B}$, which contradicts invariance of $\partial\mathcal{B}$. \square

We now proceed to define functions ϕ_{\pm} of a real variable that are key to establishing further properties of the boundary of \mathcal{B} . Denote with π_1 the projection operator on \mathbb{R}^2 given by $\pi_1(x, y) = x$. Let $I := \pi_1(\mathcal{B})$, that is, I is the set consisting of all t in \mathbb{R} for which there exists y in \mathbb{R} such that $(t, y) \in \mathcal{B}$. The set I is open in \mathbb{R} , and it has a finite or countable number of connected components (intervals). For each connected component of I choose a rational number q in the component, and label the component as I_q . Let Q be the set consisting of all such indices q . Then for each $q \in Q$, the sets I_q are open in \mathbb{R} , pairwise disjoint, and satisfy $I = \bigcup_{q \in Q} I_q$. Define for each $t \in I$,

$$\phi_{-}(t) := \inf \{y : (t, y) \in \mathcal{B}\} \quad \text{and} \quad \phi_{+}(t) := \sup \{y : (t, y) \in \mathcal{B}\}. \quad (17)$$

Note that the definition of ϕ_{\pm} implies $\text{graph}(\phi_{\pm}) \subset \partial\mathcal{B}$ and

$$\mathcal{B} = \{(t, y) \in \mathcal{R} : t \in I \text{ and } \phi_{-}(t) < y < \phi_{+}(t)\}. \quad (18)$$

Properties of ϕ_{\pm} are investigated in Claims 3–8 below.

Claim 3 *The functions ϕ_{\pm} are non-increasing on I .*

Proof. Suppose this is not the case, so there exist t_1, t_2 in I , $t_1 < t_2$, such that $\phi_+(t_1) < \phi_+(t_2)$. Choose y_1 and y_2 so that $\phi_-(t_\ell) < y_\ell < \phi_+(t_\ell)$ for $\ell = 1, 2$, and $y_2 > \phi_+(t_1)$. Then $(t_1, \phi_+(t_1))$ belongs to the order interval $[(t_1, y_1), (t_2, y_2)]$. Since $(t_\ell, y_\ell) \in \mathcal{B}$ for $\ell = 1, 2$, it follows that $(t_1, \phi_+(t_1)) \in \mathcal{B}$, which is a contradiction. Thus ϕ_+ is non-increasing on I . The proof of the corresponding statement for ϕ_- is similar. \square

For each $q \in Q$ the restriction of the function ϕ_- (resp. ϕ_+) to I_q is nonincreasing, hence it has a natural extension to the closure of I_q in the extended real line given by choosing the value at each endpoint of I_q as the one-sided limit of ϕ_- (resp. ϕ_+). We denote such extensions with ϕ_-^q and ϕ_+^q . It is a consequence of Claim 3 that for $q \in Q$, the functions ϕ_-^q and ϕ_+^q are non-increasing, and their graphs are contained in $\partial\mathcal{B}$.

Claim 4 *For every $q \in Q$, (i) $\phi_-^q(t) < \phi_+^q(t)$ for $t \in I_q$, and (ii) $\phi_-^q(t) = \phi_+^q(t)$ for $t \in \partial I_q \setminus \partial\pi_1(\mathcal{R})$.*

Proof. Statement (i) of Claim 4 follows from the definition of ϕ_\pm . To prove (ii) of Claim 4, suppose that for some $q \in Q$ and some endpoint t of I_q with $t \notin \partial\pi_1(\mathcal{R})$, the inequality $\phi_-^q(t) < \phi_+^q(t)$ holds. In this case the line segment joining $(t, \phi_-(t))$ to $(t, \phi_+(t))$ is a \preceq_{ne} -linearly ordered subset of $\partial\mathcal{B} \cap \text{int}(\mathcal{R})$, which contradicts Claim 2. Therefore $\phi_-^q(t) = \phi_+^q(t)$. \square

Claim 5 *Each of the sets $\bigcup_{q \in Q} (\text{graph}(\phi_-^q) \cap \text{int}(\mathcal{R}))$ and $\bigcup_{q \in Q} (\text{graph}(\phi_+^q) \cap \text{int}(\mathcal{R}))$ is invariant under T .*

Proof. Let $t \in \text{clos}(I)$. If $(t', y') := T(t, \phi_+(t))$, then there is a curve in \mathcal{B} with endpoint at $(t, \phi_+(t))$, so the same is true about $(t', y') := T(t, \phi_+(t))$. thus $t' \in \text{clos}(I)$. If $t' \in \partial I$, then $\phi_-(t') = \phi_+(t')$ by claim 4, so in particular $T(t, \phi_+(t)) \in$

graph ϕ_+ . Now suppose $t \in I$. For all $\delta > 0$ small enough, $(t, \phi_+(t) - \delta) \in \mathcal{B}$ and consequently $(t'_\delta, y'_\delta) := T(t, \phi_+(t) - \delta) \in \mathcal{B}$. By monotonicity of T , $(t'_\delta, y'_\delta) \ll_{ne} (t', y')$. But ϕ_- is non-increasing, so necessarily $(t', y') \in \text{graph}(\phi_+)$. \square

Claim 6 $\partial\mathcal{B} \cap \text{int}(\mathcal{R}) = \text{int}(\mathcal{R}) \cap \bigcup_{q \in Q} \text{graph}(\phi_-^q) \cup \text{graph}(\phi_+^q)$.

Claim 7 For $q \in Q$ let ϕ be either ϕ_-^q or ϕ_+^q . If $\text{graph}(\phi) \subset \text{int}(\mathcal{R})$, then ϕ is decreasing.

Proof. Arguing by contradiction, if $\phi(t_1) \leq \phi(t_2)$ for t_1, t_2 in $\text{clos}(I)$ with $t_1 < t_2$, then $T(t_1, \phi(t_1)) \leq_{ne} T(t_1, \phi(t_2))$ by strong monotonicity. The latter relation together with the invariance of $\text{graph}(\phi)$ imply that ϕ is not non-decreasing, a contradiction. \square

Claim 8 For every $q \in Q$, ϕ_-^q and ϕ_+^q are continuous on $\text{clos}(I_q)$.

Proof. Suppose ϕ_+ is not continuous at some t_0 in $\text{clos}(I)$. By the monotonic character of ϕ , the discontinuity is of the “jump” variety. More specifically, assume that ϕ_+ is defined on an interval $t_0 < t < t_0 + \delta$ for some $\delta > 0$, and $y_0 > y_+$, where $y_0 := \phi_+(t_0)$ and $y_+ := \lim_{t \rightarrow t_0^+} \phi_+(t)$. In this case, $(t_0, y_+) \in \mathcal{B}$, which is not possible. \square

Now we prove statements (i)–(iv) of Theorem 4. Let $(\alpha, \beta) := \pi_1(\mathcal{R})$ be the projection of \mathcal{B} onto the first coordinate. If \mathcal{B}' is a component of \mathcal{B} , then $\pi_1(\mathcal{B}')$ is an interval such that $I^q = \pi_1(\mathcal{B}')$ for some $q \in Q$. Define the curves \mathcal{C}_\pm by cases as follows (see Figure 8).

- (I) If $\pi_1(\mathcal{R})$ and $\pi_1(\mathcal{B}')$ have no common endpoints, \mathcal{C}_\pm is the curve given by the graph of ϕ_\pm^q .

- (II) If $\pi_1(\mathcal{R})$ and $\pi_1(\mathcal{B}')$ have β and only β as common endpoint, \mathcal{C}_- is the curve given by the graph of ϕ_-^q and \mathcal{C}_+ is the curve given by the graph of ϕ_+^q together with the line segment joining $(\beta, \phi_+^q(\beta))$ to $(\beta, \phi_-^q(\beta))$.
- (III) If $\pi_1(\mathcal{R})$ and $\pi_1(\mathcal{B}')$ have α and only α as common endpoint, \mathcal{C}_+ is the curve given by the graph of ϕ_+^q and \mathcal{C}_- is the curve given by the graph of ϕ_-^q together with the line segment joining $(\alpha, \phi_-^q(\alpha))$ to $(\alpha, \phi_+^q(\alpha))$.
- (IV) If $\pi_1(\mathcal{R})$ and $\pi_1(\mathcal{B}')$ have common endpoints α and β , \mathcal{C}_+ is the curve given by the graph of ϕ_+^q together with the line segment joining $(\beta, \phi_+^q(\beta))$ to $(\beta, \phi_-^q(\beta))$ and \mathcal{C}_- is the curve given by the graph of ϕ_-^q together with the line segment joining $(\alpha, \phi_-^q(\alpha))$ to $(\alpha, \phi_+^q(\alpha))$.

The different cases are illustrated in Figure 8. Statement (i) of Theorem 4 now follows from relation (18) and Claims 4, 7 and 8. Statement (ii) of Theorem 4 is a consequence of the order-convex character of \mathcal{B} . Since \mathcal{B}_* is connected and contains the fixed point \bar{x} , it follows $T(\mathcal{B}_*) \subset \mathcal{B}_*$. This fact and Claim 5 imply statement (iii) of Theorem 4. Now assume the hypothesis of (iv), and choose $x \in \mathcal{B}'$. Since $T^m(x) \rightarrow \bar{x}$, there exists n a positive number in \mathbb{N} such that $T^n(x) \in \mathcal{B}_*$ and $T^{n-1}(x) \notin \mathcal{B}_*$. Now statement (iv) of Theorem 4 follows from the fact that T maps connected sets to connected sets and from Claim 5. \square

Lemma 1 *Let J be a 2×2 matrix with positive entries. Let v be an eigenvector of J that is associated with the eigenvalue of J that has the smallest modulus. Let C be a closed convex double cone in \mathbb{R}^2 with vertex at the origin such that $v \notin C$. Then there exists an integer m such that $J^m(C) \subset Q_1 \cup Q_3$.*

Proof. Let λ_1 and λ_2 be eigenvalues of J , with associated eigenvectors v_1, v_2 . Assume $|\lambda_1| < \lambda_2$. We prove first that $J^m(\partial C) \subset Q_1 \cup Q_3$. If $z \in \partial C \setminus \{0\}$, then

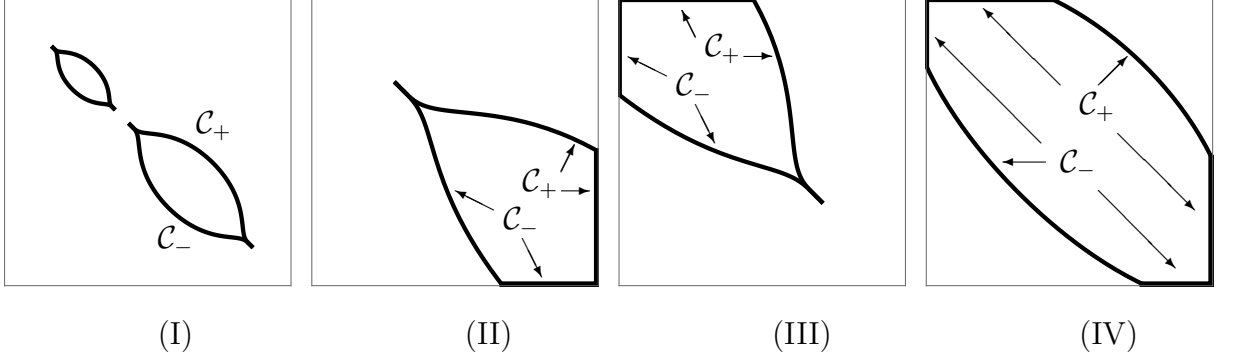


Figure 8. The four cases in the definition of \mathcal{C}_\pm .

$z = \alpha_1 v_1 + \alpha_2 v_2$ for some scalars $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_2 \neq 0$. Then

$$J^m z = \lambda_1^m \alpha_1 v_1 + \lambda_2^m \alpha_2 v_2 = \lambda_2^m \left(\left(\frac{\lambda_1}{\lambda_2} \right)^m \alpha_1 v_1 + \alpha_2 v_2 \right) \quad (19)$$

Note that v_2 has both coordinates with the same sign, by Perron-Frobenius Theorem. Since $|\frac{\lambda_1}{\lambda_2}| < 1$, it follows from (19) that for m large enough, $J^m z$ has both coordinates with sign equal to the sign of α_2 . Hence $J^m z \in Q_1 \cup Q_3$ and $J^m(\partial C) \subset Q_1 \cup Q_3$. Let \tilde{C} be the double cone in \mathbb{R}^2 that is complementary to C . Note $\tilde{C} \subset J(\tilde{C})$, similarly $\tilde{C} \subset J^m(\tilde{C})$. The set $J^m(\tilde{C})$ is a double cone with boundary in $Q_1 \cup Q_3$, and such that $v \in J^m(\tilde{C})$. Since $\mathbb{R}^2 = (J^m C) \cup (J^m \tilde{C})$, it follows that $J^m C \subset Q_1 \cup Q_3$. \square

Proof of Theorem 5. We present here a proof for the case where the point p is a fixed point of T . The case where p is a minimal period- m point may be treated by considering the map T^m , for which p is a fixed point, and it is not given here. By Theorem 4, $\mathcal{B} = \mathcal{C}_- \cup \mathcal{C}_+$. Without loss of generality we assume $p \in \mathcal{C}_+$. If the conclusion is not true, then there exists a double cone C containing $\ell \setminus \{p\}$ in its interior and a sequence $\{x_m\}$ on the curve \mathcal{C}_+ such that $x_m \rightarrow p$ and $x_m \notin C$. Choose an integer m as in Lemma 1, and set $J^m := \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$. Let $\{(t, \phi(t)) : t \in I\}$

be a parametrization of \mathcal{C}_+ near p , such that for some t_* in the real interval I , $(t_*, \phi(t_*)) = p$. Set

$$L_n := \left(\tilde{a}(t_n - t_*) + \tilde{b}(\phi(t_n) - \phi(t_*)), \tilde{c}(t_n - t_*) + \tilde{d}(\phi(t_n) - \phi(t_*)) \right) \quad (20)$$

From Lemma 1, $L_n \in Q_1 \cup Q_3$. Also, $L_n \neq (0, 0)$, since the null space of \tilde{J} , if nontrivial, is contained in the cone C . Then,

$$\text{for each } n \geq 0, L_n \text{ has nonzero coordinates with the same sign.} \quad (21)$$

Let K be a compact interval containing t_* in its interior and define, for $t \in K$ and $\epsilon_1, \epsilon_2 \in [-1, 1]$,

$$\begin{aligned} \Phi_1(t, \epsilon_1) &:= \left((\tilde{a} + \epsilon_1)(t - t_*) + (\tilde{b} - \epsilon_1)(\phi(t) - \phi(t_*)) \right) \\ \Phi_2(t, \epsilon_2) &:= \left((\tilde{c} + \epsilon_2)(t - t_*) + (\tilde{d} - \epsilon_2)(\phi(t) - \phi(t_*)) \right) \end{aligned} \quad (22)$$

The functions Φ_1 and Φ_2 are uniformly continuous on $K \times [-1, 1]$, and by (21), $\Phi_1(t, 0)$ and $\Phi_2(t, 0)$ are nonzero and have the same sign for all n . By uniform continuity, there exists $\delta > 0$ such that $\Phi_1(t, \epsilon_1)$ and $\Phi_2(t, \epsilon_2)$ are nonzero and have the same sign for all $t \in K$ and $|\epsilon_1| < \delta$, $|\epsilon_2| < \delta$. Without loss of generality we may assume that

$$t_n > t_* \quad \text{and} \quad \phi(t_n) < \phi(t_*), \quad n \in \mathbb{N}. \quad (23)$$

Let J and \tilde{J} be the jacobian matrices of T and T^m at p respectively. Since p is a fixed point, the chain rule gives $\tilde{J} = J^m$. Note the entries of \tilde{J} are positive. Set

$\tilde{J} := \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$. For each $n \in \mathbb{N}$ define $o_n^{(1)}$ and $o_n^{(2)}$ by

$$(o_n^{(1)}, o_n^{(2)}) := T^m(t_n, \phi(t_n)) - T^m(t_*, \phi(t_*)) - L_n. \quad (24)$$

Since T is continuously differentiable,

$$g_n^{(\ell)} := \frac{o_n^{(\ell)}}{|t_n - t_*| + |\phi(t_n) - \phi(t_*)|} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad \ell = 1, 2. \quad (25)$$

Rearranging terms in (24) and using (20), (23) and (25) we have

$$\begin{aligned} & T^m(t_n, \phi(t_n)) - T^m(t_*, \phi(t_*)) \\ &= \left((\tilde{a} + g_n^{(1)})(t_n - t_*) + (\tilde{b} - g_n^{(1)})(\phi(t_n) - \phi(t_*)), \tilde{c} + g_n^{(2)} + (\tilde{d} - g_n^{(2)})(\phi(t_n) - \phi(t_*)) \right) \end{aligned} \quad (26)$$

By (21), (23) and (25) and the assumption that \tilde{a} , \tilde{b} , \tilde{c} and \tilde{d} are positive, both coordinates in the right-hand side of (26) have the same sign for large n , and therefore either $T(t_n, \phi(t_n)) \preceq T(t_*, \phi(t_*))$ or $T(t_*, \phi(t_*)) \preceq T(t_n, \phi(t_n))$. But this contradicts (i) of Theorem 4, which requires points on \mathcal{C}_+ to be non-comparable.

□

Proof of Theorem 6. (i) Let $p \in \mathcal{C}_+$, and let $\{(t, \phi(t)) : t \in I\}$ be a parametrization of \mathcal{C}_+ near p , such that for some $t_* \in I$, $(t_*, \phi(t_*)) = p$. Here $I \subset \mathbb{R}$ is an interval. The function ϕ is decreasing. If ϕ is not Lipschitz at t_* , then there exists a sequence $\{t_n\}$ in I such that $t_n \rightarrow t_*$ and

$$\left| \frac{\phi(t_n) - \phi(t_*)}{t_n - t_*} \right| \rightarrow \infty \quad \text{as } n \rightarrow \infty. \quad (27)$$

Without loss of generality we may assume that $t_n > t_*$ and $\phi(t_n) < \phi(t_*)$ for all n , that is,

$$t_n \downarrow t_* \quad \text{and} \quad \frac{\phi(t_n) - \phi(t_*)}{t_n - t_*} \rightarrow -\infty \quad \text{as } n \rightarrow \infty \quad (28)$$

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the jacobian matrix of T at p . For each $n \in \mathbb{N}$ define $o_n^{(1)}$ and $o_n^{(2)}$ by

$$\begin{aligned} (o_n^{(1)}, o_n^{(2)}) := & T(t_n, \phi(t_n)) - T(t_*, \phi(t_*)) \\ & - (a(t_n - t_*) + b(\phi(t_n) - \phi(t_*)), c(t_n - t_*) + d(\phi(t_n) - \phi(t_*))) \end{aligned} \quad (29)$$

Since T is continuously differentiable,

$$g_n^{(\ell)} := \frac{o_n^{(\ell)}}{|t_n - t_*| + |\phi(t_n) - \phi(t_*)|} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad \ell = 1, 2. \quad (30)$$

Rearranging terms in (29) and using (28) and (30) we have

$$\begin{aligned} & T(t_n, \phi(t_n)) - T(t_*, \phi(t_*)) \\ &= (t_n - t_*) \left(a + g_n^{(1)} + (b - g_n^{(1)}) \frac{\phi(t_n) - \phi(t_*)}{t_n - t_*}, c + g_n^{(2)} + (d - g_n^{(2)}) \frac{\phi(t_n) - \phi(t_*)}{t_n - t_*} \right). \end{aligned} \quad (31)$$

By (28) and (30) and the assumption that a, b, c and d are positive, both coordinates in the right-hand side of (31) are negative for large n , and therefore $T(t_n, \phi(t_n)) \preceq T(t_*, \phi(t_*))$. But this contradicts (i) of Theorem 4, which requires points on \mathcal{C}_+ to be non-comparable. Thus ϕ is Lipschitz.

(ii) We present the proof for the case when p is a fixed point of T . Note p is necessarily unstable since $p \in \partial\mathcal{B}$. Since it is hyperbolic, it is either a saddle point or a source. If p is a saddle point, then it has a local stable manifold M^s , which is tangential to v with v not comparable to the origin by the Krein-Rutman theorem. There exist points x in \mathcal{B}_* that are arbitrarily close to p and which belong to the union of quadrants $Q_2(p)$ and $Q_4(p)$. Furthermore, such points x may be chosen to be comparable to points on M^s , which would prevent the iterates of such points from converging to p , thus contradicting the definition of stable manifold. \square

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MANUSCRIPT 3

Global Dynamics Results for Discrete Planar Cooperative Maps

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Abstract

Sufficient conditions are given for planar cooperative maps to have the qualitative global dynamics determined solely on local stability information obtained from fixed and minimal period-two points. The results are given for a class of strongly cooperative planar maps of class C^1 on an order interval. The maps are assumed to have a finite number of strongly ordered fixed points, and also the minimal period-two points are ordered in a sense. An application is included.

3.1 Introduction

In this paper we consider a cooperative system of the form

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n), \quad n = 0, 1, \dots, \quad (32)$$

where the transition functions f, g are non-decreasing in all its arguments and its corresponding map $F = (f, g)$. Sufficient conditions are given for such planar cooperative map to have the qualitative global dynamics determined by local stability information obtained from fixed and minimal period-two points. The results are given for a class of strongly cooperative planar maps of class C^1 defined on an order interval. The maps are assumed to have a finite number of strongly ordered fixed points and minimal period-two points. Our results holds in hyperbolic case as well as in some non-hyperbolic cases as well. Our results are motivated by global dynamic results for the systems

$$\begin{aligned} x_{n+1} &= a x_n + \frac{b y_n}{\delta + y_n} \\ y_{n+1} &= \frac{c x_n}{\delta + x_n} + d y_n, \quad n = 0, 1, \dots \end{aligned} \quad (33)$$

and

$$\begin{aligned} x_{n+1} &= a x_n + \frac{b y_n^2}{\delta + y_n^2} \\ y_{n+1} &= \frac{c x_n^2}{\delta + x_n^2} + d y_n, \quad n = 0, 1, \dots \end{aligned} \tag{34}$$

see [3, 4, 5, 17]. System (33) was considered in [3] and it was proved that all bounded solutions exhibited global attractivity to either the zero equilibrium or to the unique positive equilibrium. More precisely, it was shown in [3] the global dynamics of system (33) where $a, b, c, d, \delta > 0$, $x_0, y_0 \geq 0$ is simple and can be described in terms of bifurcation theory as the transcritical bifurcation which causes an exchange of stability when $(1 - a)(1 - d)\delta^2 - bc$ passes through the value 0.

The system (34), studied in [4] exhibited the appearance of period-two solutions, which played an important role in the global dynamics of this system. Cases were provided for such system which had 1, 2 or 3 period-two solutions and in the last case one of these period-two solutions had substantial basin of attraction [4]. Papers [3, 4] extensively used the algebraic techniques to find the regions of existence and stability of equilibrium solutions and period-two solutions. The results in this paper will be proven through the geometric analysis of the equilibrium curves and by using some major results about the global stable and unstable manifolds of cooperative systems in the plane [21]-[24]. The results of this paper are applicable to systems (33) and (34). Our results have immediate extension to the competitive systems of difference equations in the plane.

The paper is organized as follows. In Section 3.2 we list some basic results that are relevant to this paper, see [21]-[24]. See also [1, 2, 6, 9, 10, 11, 12, 15, 16, 19, 20, 25, 26, 27, 28] for some related competitive systems. In Section 3.3 we present the three main theorems. In section 3.4 we apply the theorems from Section 3.3 to a parametrized cooperative system whose transition functions are of the Holling's type [17]. Finally the proofs of the results in Section 3.3 are presented

in Section 3.5.

3.2 Preliminaries

Let \preceq be a partial order on \mathbb{R}^n with nonnegative cone P . For $x, y \in \mathbb{R}^n$ the *order interval* $\llbracket x, y \rrbracket$ is the set of all z such that $x \preceq z \preceq y$. We say $x \prec y$ if $x \preceq y$ and $x \neq y$, and $x \ll y$ if $y - x \in \text{int}(P)$. A map T on a subset of \mathbb{R}^n is *order preserving* if $T(x) \preceq T(y)$ whenever $x \prec y$, *strictly order preserving* if $T(x) \prec T(y)$ whenever $x \prec y$, and *strongly order preserving* if $T(x) \ll T(y)$ whenever $x \prec y$.

Let $T : R \rightarrow R$ be a map with a fixed point \bar{x} and let R' be an invariant subset of R that contains \bar{x} . We say that \bar{x} is *stable* (*asymptotically stable*) relative to R' if \bar{x} is a *stable* (*asymptotically stable*) fixed point of the restriction of T to R' . The basin of attraction of a fixed point \bar{x} , denoted as $\mathcal{B}(\bar{x})$ is defined as $\mathcal{B}(\bar{x}) = \{y : T^n(y) \rightarrow \bar{x}\}$. Subsolution (resp. supersolution) for the map T is a point which satisfies $x \preceq T(x)$ (resp. $T(x) \preceq x$). A fixed point $u \in V$ is said to be *order stable from below* if there exists a strictly increasing sequence of subsolutions v_n in V convergent to u . A fixed point $u \in V$ is said to be *strongly order stable from below* if there exists a strictly increasing sequence of strict subsolutions v_n in V convergent to u . The notions of *order stable from above* and *strongly order stable from above* are defined similarly. A *(strongly) order stable fixed point* has the respective stability property from above and below [7].

Throughout this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the *South-East* (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East (NE) ordering is called *cooperative* and a map monotone with respect to the South-East (SE) ordering is called *competitive*. A map T on a nonempty

set $\mathcal{R} \subset \mathbb{R}^2$ which second iterate T^2 is monotone with respect to the North-East (resp. South-East) ordering is called *anti-cooperative* (resp. *anti-competitive*).

If T is differentiable map on a nonempty set \mathcal{R} , a sufficient condition for T to be strongly monotone with respect to the NE ordering is that the Jacobian matrix at all points x has the sign configuration

$$\text{sign}(J_T(\mathbf{x})) = \begin{bmatrix} + & + \\ + & + \end{bmatrix}, \quad (35)$$

provided that \mathcal{R} is open and convex.

The next result in [24] is stated for order-preserving maps on \mathbb{R}^n . See [14] for a more general version valid in ordered Banach spaces.

Theorem 7 *For a nonempty set $R \subset \mathbb{R}^n$ and \preceq a partial order on \mathbb{R}^n , let $T : R \rightarrow R$ be an order preserving map, and let $a, b \in R$ be such that $a \prec b$ and $\llbracket a, b \rrbracket \subset R$. If $a \preceq T(a)$ and $T(b) \preceq b$, then $\llbracket a, b \rrbracket$ is an invariant set and*

- i.) *There exists a fixed point of T in $\llbracket a, b \rrbracket$.*
- ii.) *If T is strongly order preserving, then there exists a fixed point in $\llbracket a, b \rrbracket$ which is stable relative to $\llbracket a, b \rrbracket$.*
- iii.) *If there is only one fixed point in $\llbracket a, b \rrbracket$, then it is a global attractor in $\llbracket a, b \rrbracket$ and therefore asymptotically stable relative to $\llbracket a, b \rrbracket$.*

The following result is a direct consequence of the Trichotomy Theorem, see [14, 24], and is helpful for determining the basins of attraction of the equilibrium points.

Corollary 4 *If the nonnegative cone of a partial ordering \preceq is a generalized quadrant in \mathbb{R}^n , and if T has no fixed points in $\llbracket u_1, u_2 \rrbracket$ other than u_1 and u_2 , then the*

interior of $\llbracket u_1, u_2 \rrbracket$ is either a subset of the basin of attraction of u_1 or a subset of the basin of attraction of u_2 .

Next result is a simple and useful geometric test for checking when the fixed point of the cooperative map is non-hyperbolic.

Lemma 2 *Let (\bar{x}, \bar{y}) be an interior fixed point of a cooperative map $R(x, y) = (f(x, y), g(x, y))$, and let r be the spectral radius of the Jacobian matrix $J_R(\bar{x}, \bar{y})$. Suppose the tangent lines to $f(x, y) = x$ and $g(x, y) = y$ at (\bar{x}, \bar{y}) are not parallel to one of the axes. Denote with m_1 and m_2 respectively the slopes of the tangent lines. The following statements are true:*

- (i) *If $0 < m_2 < m_1$, then $r < 1$.*
- (ii) *If $0 < m_1 = m_2$, then $r = 1$.*
- (iii) *If $0 < m_1 < m_2$, then $r > 1$.*

Proof. Without loss of generality assume $(\bar{x}, \bar{y}) = (0, 0)$. Let $J = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be the jacobian of R at the origin. Note the tangent lines to $f(x, y) = x$ and $g(x, y) = y$ at $(0, 0)$ are given by $\alpha x + \beta y = x$ and $\gamma x + \delta y = y$. Thus the entries of J are nonzero, and m_1 and m_2 are respectively the slopes of the lines $\alpha x + \beta y = x$ and $\gamma x + \delta y = y$. Since $m_1 = (1 - \alpha)/\beta$ and $m_2 = \gamma/(1 - \delta)$, from $m_1 > 0$ and $m_2 > 0$, we obtain $0 < \alpha < 1$ and $0 < \delta < 1$. The characteristic polynomial of J , $p(t) := t^2 - (a + d)t + ad - bc$, has real and distinct roots s and r , with $|s| < r$. Then,

$$m_1 - m_2 = \frac{1 - \alpha}{\beta} - \frac{\gamma}{1 - \delta} = \frac{1 - \alpha - \delta + \alpha\delta - \beta\gamma}{\beta(1 - \delta)} = \frac{p(1)}{\beta(1 - \delta)} \quad (36)$$

Note the minimum of $y = p(t)$ is attained at $t = \frac{\alpha + \delta}{2} < 1$. If $m_1 > m_2$, then $p(1) > 0$, which implies $r < 1$. If $m_1 = m_2$, then $p(1) = 0$, hence $r = 1$. Similarly, if $m_1 < m_2$, then $p(1) < 0$, so $r > 1$ in this case. \square

3.3 Main Results

In this section we present three theorems. Theorem 8 gives a qualitative characterization of global dynamics of a class of bounded planar cooperative maps. The maps are assumed to have hyperbolic fixed points that are strongly ordered and to have no minimal period-two points. A result for maps with a non-hyperbolic fixed point is considered in Theorem 9. If minimal period-two points are present, Theorem 10 gives information that in many situations is sufficient to produce a description of the global dynamics of the map.

Suppose $\mathcal{R} = [a, b]$ is an order interval in \mathbb{R}^2 , and $T : \mathcal{R} \rightarrow \mathcal{R}$ is a given cooperative map. When the number of fixed points of T is one or two, global dynamics of T can be determined from basic properties of monotone maps and the Trichotomy Theorem [15]. Indeed, since T is continuous and \mathcal{R} is compact and connected, T must have a fixed point. If the fixed point is unique, by monotonicity of T it is a global attractor. Now suppose T has exactly two fixed points x and y such that $x \prec y$. Then a is a supersolution and b is a subsolution, and therefore $\bigcap_{n=0}^{\infty} T^n([a, b]) = [x, y]$. The Trichotomy Theorem implies that one fixed point is order stable and the other one is unstable, with the interior of $[x, y]$ being attracted to either x or y . For bounded strongly cooperative maps with three or more hyperbolic strongly ordered fixed points and with no minimal period-two points, the following result gives a qualitative global dynamics description.

Theorem 8 *Let \mathcal{R} be an order interval (with respect to the north-east ordering) in \mathbb{R}^2 . Let $T : \mathcal{R} \rightarrow \mathcal{R}$ be a map that is strongly cooperative, bounded, and of class C^1 in the interior of \mathcal{R} . Assume (H1) The set \mathcal{F} of fixed points of T satisfies $3 \leq |\mathcal{F}| < \infty$, (H2) \mathcal{F} is strongly ordered, and (H3) $\mathcal{F} \cap \text{int}(\mathcal{R})$ consists only of hyperbolic fixed points.*

If T has no minimal period-two points, then there exists an integer k with

$1 \leq k \leq |\mathcal{F}| - 2$, and there exist k invariant strongly south-east ordered pairwise disjoint curves $\mathcal{C}_1, \dots, \mathcal{C}_k$ in \mathcal{R} such that \mathcal{C}_ℓ has endpoints in $\partial\mathcal{R}$, \mathcal{C}_ℓ contains a fixed point of T and only one. Every orbit in \mathcal{C}_ℓ converges to the fixed point in \mathcal{C}_ℓ , which is a saddle fixed point. Each connected component of $\mathcal{R} \setminus \cup\{\mathcal{C}_1, \dots, \mathcal{C}_k\}$ is the basin of attraction of an order stable fixed point.

Corollary 5 *If T has no interior repelling fixed points, then every orbit converges to a fixed point.*

The following result is useful in the study of planar cooperative maps with non-hyperbolic fixed points.

Theorem 9 *For $a, b \in \mathbb{R}^2$ with $a \prec b$, let $T : [a, b] \rightarrow [a, b]$ be a strongly cooperative map that is of class C^1 in the interior of $[a, b]$ and such that*

(H1) *a and b are fixed points of T , and $[a, b]$ contains a unique interior fixed point c .*

(H2) *a and c are order stable from above or b and c are order stable from below.*

(H3) *There are no minimal period-two points in $[a, b]$.*

Then there exists a strongly south-east ordered invariant Lipschitz curve \mathcal{C} through c and with endpoints on the boundary of $[a, b]$, such that each of the two connected components of $[a, b] \setminus \mathcal{C}$ is a subset of the basin of attraction of a fixed point. Also, for $x \in \mathcal{C}$, $T^n(x) \rightarrow c$.

Remark 6 By hypothesis (H1) of Theorem 9, in (H3) it is enough to require $x \in Q_2(c) \cap Q_4(c)$. Also, hypothesis (H2) implies that the spectral radius of the jacobian matrix of T at c equals 1. In particular, c is non-hyperbolic.

If a bounded strongly cooperative map T with $T : \mathcal{R} \rightarrow \mathcal{R}$ has a minimal period-two point p , then there exist fixed points a and b such that p is in the interior of $[a, b]$. Indeed, just choose x and y in \mathcal{R} such that $x \preceq z \preceq y$ for all $y \in T(\mathcal{R})$. Then $x \preceq T(x)$ and $T(y) \preceq y$, that is, x is a super solution and y is a sub solution. Both have bounded iterates that satisfy $T^n(x) \prec p \prec T^n(y)$, for $n = 1, 2, \dots$. Such iterates must converge to fixed points a and b such that $a \prec p \prec b$. The next result implies that, under hypotheses that include (among others) the non-existence of minimal period-four points and the existence of a unique interior fixed point, the global dynamics picture of T on $[a, b]$ is quite simple.

Theorem 10 *For $a, b \in \mathbb{R}^2$ with $a \prec b$, let $T : [a, b] \rightarrow [a, b]$ be a strongly cooperative map that is of class C^1 in the interior of $[a, b]$ and such that*

- (H1) *a and b are order stable fixed points of T , and $[a, b]$ contains a unique interior fixed point c ,*
- (H2) *There are no minimal period-four points in $[a, b]$.*
- (H3) *If $x \in [a, b]$ satisfies $T(x) = c$, then $x = c$.*
- (H4) *The smaller characteristic value of T at each fixed point or minimal period-two point in $[a, b]$ is not -1 .*

Then the following statements are true.

- (i) *If $\{p_1, T(p_1)\}$ is a unique minimal period two orbit in $[a, b]$, then c is a repeller and p_1 is a periodic saddle point. The basins of a and b in $[a, b]$ have a common boundary in the interior of $[a, b]$ which is a strongly south-east ordered invariant curve \mathcal{C}_1 that contains $\{p_1, T(p_1)\}$ and c , and that has endpoints in the boundary of $[a, b]$. If $x \in \mathcal{C}_1$ satisfies $x \neq c$, then $T^n(x)$ is attracted to $\{p_1, T(p_1)\}$.*

(ii) If $\{p_1, T(p_1)\}$ and $\{p_2, T(p_2)\}$ are the only minimal period two orbits in $[a, b]$ and if $p_1 \prec p_2$, then the boundary of the basin of a in the interior of $[a, b]$ is a strongly south east ordered invariant curve \mathcal{C}_1 that contains $\{p_1, T(p_1)\}$ and c , and the boundary of the basin of b in the interior of $[a, b]$ is a strongly south east ordered invariant curve \mathcal{C}_2 that contains $\{p_2, T(p_2)\}$ and c , both \mathcal{C}_1 and \mathcal{C}_2 have endpoints on the boundary of $[a, b]$, and $\mathcal{C}_1 \cap \mathcal{C}_2 = \{c\}$. The point c is a repelling fixed point, one of p_1, p_2 minimal period-two periodic point is a saddle, and the other is a non-hyperbolic semistable periodic point. If $x \in \mathcal{C}_1$ satisfies $x \neq c$, then $T^n(x)$ is attracted to $\{p_1, T(p_1)\}$. If $x \in \mathcal{C}_2$ satisfies $x \neq c$, then $T^n(x)$ is attracted to $\{p_2, T(p_2)\}$. The region bounded \mathcal{R}_1 bounded by \mathcal{C}_1 and \mathcal{C}_2 is invariant. If $x \in \mathcal{R}_1$, then $T^n(x)$ is attracted to the orbit of the non-hyperbolic periodic point.

(iii) Suppose T has exactly $k \geq 3$ minimal period-two orbits $\{p_1, T(p_1)\}, \dots, \{p_k, T(p_k)\}$ where p_1, \dots, p_k are hyperbolic and $p_1 \prec p_2 \prec \dots \prec p_k$. Then c is a repelling fixed point, k is odd, and for $1 \leq \ell \leq k$, p_ℓ is a periodic saddle if ℓ is odd, while p_ℓ is LAS if ℓ is even. There exist strongly south east ordered invariant curves $\mathcal{C}_1, \dots, \mathcal{C}_{\frac{1}{2}(k+1)}$ with endpoints on the boundary of $[a, b]$ such that for $i, j \in \{1, \dots, \frac{1}{2}(k+1)\}$ with $i \neq j$, $\mathcal{C}_i \cap \mathcal{C}_j = \{c\}$, $\{p_{2i-1}, T(p_{2i-1})\} \subset \mathcal{C}_i$, and $\{p_{2i}, T(p_{2i})\}$ is a subset of the open region \mathcal{R}_i bounded by \mathcal{C}_{2i-1} and \mathcal{C}_{2i+1} . For every $x \in \mathcal{C}_i$ that satisfies $x \neq c$, $T^n(x)$ is attracted to $\{p_{2i-1}, T(p_{2i-1})\}$. For every x in \mathcal{R}_i , $T^n(x)$ is attracted to $\{p_{2i}, T(p_{2i})\}$. The region \mathcal{R}_i is invariant. The boundary of the basin of a (respectively, b) in the interior of $[a, b]$ is \mathcal{C}_1 (resp. \mathcal{C}_k).

Remark 7 Suppose in Theorem 10 the map T has a bounded and smooth strongly cooperative extension \tilde{T} on a domain \mathcal{R} which contains $[a, b]$ so that a and b are locally asymptotically stable, and such that fixed points are strongly ordered

and all minimal period-two points are contained in $[a, b]$. Assume also \tilde{T} has no minimal period-four points in \mathcal{R} , and that the equation $T(x) = c$ has $x = c$ as its only solution in \mathcal{R} . By Theorem 4 in [18] there exist south-east strongly ordered curves $\mathcal{C}_+(a)$ through a and $\mathcal{C}_-(b)$ through b that are part of the boundary of the basin of a and b respectively, and so that $c \in \mathcal{C}_+(a) \cap \mathcal{C}_-(b)$. Thus necessarily $\mathcal{C} \subset \mathcal{C}_+(a) \cap \mathcal{C}_-(b)$. Now endpoints of both $\mathcal{C}_+(a)$ and $\mathcal{C}_-(b)$ belong to the boundary of \mathcal{R} , since otherwise any such endpoint is a fixed point or minimal period-two point, by Corollary 2 in [18]. This is not possible because of the assumptions on \tilde{T} . Boundedness of \tilde{T} can now be used to prove that points on both $\mathcal{C}_+(a)$ and $\mathcal{C}_-(b)$ have iterates that converge to a minimal period-two point. Finally, we prove $\mathcal{C}_+(a) = \mathcal{C}_-(b)$. If $\mathcal{C}_+(a)$ and $\mathcal{C}_-(b)$ are not the same curve, then points x and y can be chosen in $\mathcal{C}_+(a)$ and $\mathcal{C}_-(b)$ respectively so that $x \prec y$. Then for all $n \geq 1$, $T^n(x) \prec T^n(y)$. But for n large enough, $T^n(x)$ and $T^n(y)$ both belong to \mathcal{C} , which is strongly ordered in the southeast order. This contradiction completes the argument.

3.4 Global dynamics of a cooperative system

The purpose of this section is to illustrate the application of the results in this paper. Consider the following parametrized system of difference equations of Holling type:

$$\begin{aligned} x_{n+1} &= \frac{a x_n}{\delta_1 + x_n} + \frac{b y_n^2}{\delta_2 + y_n^2}, \\ y_{n+1} &= \frac{c x_n^2}{\delta_2 + x_n^2} + \frac{d y_n}{\delta_1 + y_n}, \end{aligned} \quad n = 0, 1, 2, \dots \quad (37)$$

where $a, b, c, d, \delta_1, \delta_2 > 0$, $x_0, y_0 \geq 0$. Let $T : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ be the map associated to (37), that is

$$T(x, y) = \left(\frac{a x}{\delta_1 + x} + \frac{b y^2}{\delta_2 + y^2}, \frac{c x^2}{\delta_2 + x^2} + \frac{d y}{\delta_1 + y} \right). \quad (38)$$

The results in Section 3.3 imply that if certain properties of the map in question are satisfied, then the qualitative global dynamics pictures of T can be deduced from the study of fixed and periodic points. We now proceed to establish properties of the parametrized family of maps.

Proposition 2 *For $a, b, c, d, \delta_1, \delta_2 > 0$, let $e_0 := (0, 0)$ and $u := (\frac{a}{\delta_1} + \frac{b}{\delta_2}, \frac{c}{\delta_2} + \frac{d}{\delta_1})$. Then the map T in (38) is bounded and satisfies $T(\mathbb{R}_+^2) \subset [e_0, u]$, T is strongly monotonic on its domain and it is smooth on a neighborhood of \mathbb{R}_+^2 . The set \mathcal{F} of fixed points of T is strongly ordered, finite, and contains the origin.*

Proof. The jacobian matrix of T at (x, y) is

$$J_T(x, y) = \begin{pmatrix} \frac{a \delta_1}{(x + \delta_1)^2} & \frac{2 b \delta_2}{(y^2 + \delta_2)^2} \\ \frac{2 c x \delta_2}{(x^2 + \delta_2)^2} & \frac{d \delta_1}{(y + \delta_1)^2} \end{pmatrix} \quad (39)$$

Since $J_T(x, y)$ has positive entries, T is strongly monotonic on \mathbb{R}_+^2 . The increasing character of the coordinate entries of T with respect to each variable gives that T is bounded with range $[0, \frac{a}{\delta_1} + \frac{b}{\delta_2}) \times [0, \frac{c}{\delta_2} + \frac{d}{\delta_1})$. By direct substitution in (38) we have the origin e_0 is a fixed point for all values of the parameters. Fixed points of T are common points of the *equilibrium curves*

$$(C_1) : \quad x = \frac{a x}{\delta_1 + x} + \frac{b y^2}{\delta_2 + y^2} \quad (40)$$

$$(C_2) : \quad y = \frac{c x^2}{\delta_2 + x^2} + \frac{d y}{\delta_1 + y} \quad (41)$$

It is obvious that the origin is a fixed point. Since (40) and (41) may be written as polynomial (quartic) equations, Bezout's theorem gives that there are at most 16 fixed points. For (x, y) in the positive quadrant, (40) and (41) may be rewritten as

$$y = \sqrt{\frac{-\delta_2 x (x + \delta_1 - a)}{x^2 + (\delta_1 - a - b)x - b\delta_1}}, \quad \max(a - \delta_1, 0) < x < A, \quad (42)$$

$$x = \sqrt{\frac{-\delta_2 y (y + \delta_1 - d)}{y^2 + (\delta_1 - c - d)y - c\delta_1}}, \quad \max(d - \delta_1, 0) < y < B, \quad (43)$$

where $A := \frac{1}{2} \left(a + b - \delta_1 + \sqrt{(a + b - \delta_1)^2 + 4b\delta_1} \right)$ and $B := \frac{1}{2} \left(c + d - \delta_1 + \sqrt{(c + d - \delta_1)^2 + 4c\delta_1} \right)$. Some basic calculations that we skip show that (42) defines y as an increasing function of x in $[0, A)$, and (43) defines x as an increasing function of y in $[0, B)$. Thus the set \mathcal{F} of fixed points of T is linearly ordered in the north-east order, and $\mathcal{F} \subset [0, A) \times [0, B)$. \square

Proposition 2 establishes that the parametrized family of maps (38) satisfies some of the hypotheses of the results in Section 3.3. The remaining hypotheses concern fixed and periodic points. In the present case, number and type of fixed and periodic points naturally depends on the choice of parameters that appear in the definition of the map. Equations (40) and (41) are to be solved to obtain fixed points, but these equations do not lend themselves to a simple criterion for classifying fixed points; the number of parameters is just too high and the polynomial equations that can be obtained have a high degree.

Numerical searches performed by the authors of this article suggest that four is the maximum number of fixed points for system (37), and that three is the maximum number of minimal period-two orbits.

In the rest of this section we illustrate the application of the results of this paper for different parameter choices given in Table 1. The different cases are

presented in Figures 9 – 14. For specific values of the map T , fixed points and minimal period-two points can be easily found with a computer algebra system (CAS). Also a CAS can be used to determine that a specific fixed point has only one pre-image. CAS do not work to investigate existence of minimal period-four points algebraically due to the complexity of the equations involved. In this case one can use other means such as the approach mentioned in Figure 15.

Case	a	b	δ_1	c	d	δ_2
(1)	0.345	2	1.585	1.04	0.435	0.73
(2)	0.500	2	1.585	0.65	0.435	0.73
(3)	0.375	2.	0.98	0.505	1.098	0.73
(4)	0.347447	5.26514	4.54021	0.774986	0.137048	9.06053
(5)	0.277479	8.91088	0.115188	3.74014	0.4013	7.43602
(6)	0.277479	8.91088	3.39209	0.331034	0.115188	7.43602

Table 1. Parameter values used in Figs. 9 – 14

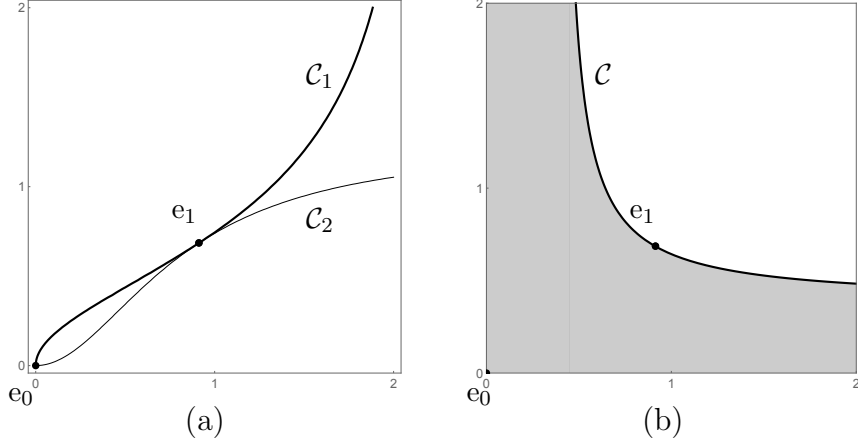


Figure 9. Case 1 in Table 1. There exists a unique interior fixed point e_1 , which is non-hyperbolic and stable from above. The origin e_0 is stable from above. There are no minimal period-two points. (a) shows the equilibrium curves, which have a tangential contact point at e_1 , as implied by Lemma 2. (b) By Theorem 9 applied to the restriction of the map to the invariant order interval $[e_0, u]$, there exists a southeast ordered curve \mathcal{C} through e_1 , such that points below \mathcal{C} are attracted to e_0 and points on or above \mathcal{C} are attracted to e_1 . Since there are no period-two points outside $[e_0, u]$, the curve \mathcal{C} has an extension to a southeast ordered curve on \mathbb{R}_+^2 with endpoints on the boundary which separates the basins of attraction of e_0 and e_1 .

3.5 Proofs of Theorems

Proof of Theorem 8. Denote with \bar{x}_ℓ , $\ell = 1, \dots, m$ the fixed points of T ordered so that $\bar{x}_\ell \prec \bar{x}_{\ell+1}$, for $1 \leq \ell \leq m-1$. Choose $\ell \in \{1, \dots, m-1\}$. Then $[\bar{x}_\ell, \bar{x}_{\ell+1}]$ has only two fixed points, so by the Trichotomy Theorem one of the fixed points is stable and the other is unstable. Thus either all even indexed fixed points are stable, or all odd indexed fixed points are stable. By boundedness of T , interior unstable fixed points belong to an order interval in \mathcal{R} determined by two stable fixed points. Suppose \bar{x}_ℓ and $\bar{x}_{\ell+2}$ are stable, and $\bar{x}_{\ell+1}$ is unstable. By Theorem 4 and Theorem 6 in [18] there exist south-east ordered Lipschitz curves $\mathcal{C}_+^\ell, \mathcal{C}_-^\ell$, such that $\partial\mathcal{B}(\bar{x}_\ell) = \mathcal{C}_+^\ell \cup \mathcal{C}_-^\ell$, and whose only possible common points are endpoints, and in this case such points are either fixed points or minimal period-two points. Neither of those two possibilities is allowed by the hypotheses of the Theorem, hence endpoints do not coincide. Note that $\bar{x}_{\ell+1} \in \mathcal{C}_+^\ell$. Thus the dynamics of T

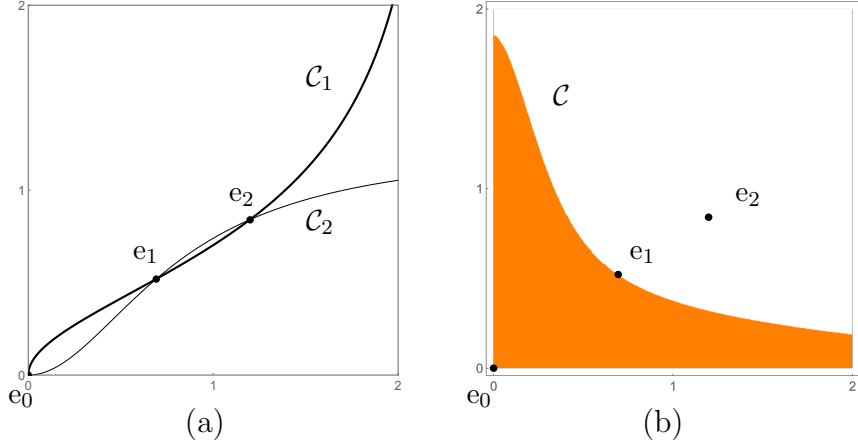


Figure 10. Case 2 in Table 1. There exist interior fixed points e_1 (unstable) and e_2 (stable). The origin e_0 is stable. There are no minimal period-two points. (a) shows the equilibrium curves, which have a nontangential contact points at e_1 and e_2 . Either a calculation or Lemma 2 may be used to determine local stability of e_1 and e_2 . (b) By Theorem 9 applied to the restriction of the map to the invariant order interval $[e_0, e_2]$, there exists a southeast ordered curve \mathcal{C} through e_1 , such that points below \mathcal{C} are attracted to e_0 points above \mathcal{C} are attracted to e_2 , and points on \mathcal{C} are attracted to e_1 . Since there are no period-two points outside $[e_0, e_2]$, the curve \mathcal{C} has an extension to a southeast ordered curve on \mathbb{R}_+^2 with endpoints on the boundary which separates the basins of attraction of e_0 and e_2 .

on the curve \mathcal{C}_+^ℓ and \mathcal{C}_-^ℓ is one-dimensional, bounded, with only one fixed point, namely $\bar{x}_{\ell+1}$ and no minimal period-two points. By Theorem C.3 in [8], the iterates of each point on \mathcal{C}_+^ℓ must converge to a fixed point. Such point can only be $\bar{x}_{\ell+1}$. We conclude that $\bar{x}_{\ell+1}$ is a saddle point. A similar argument can be made with the point $\bar{x}_{\ell+1}$ and the curve $\mathcal{C}_-^{\ell+2}$. Thus \mathcal{C}_+^ℓ and $\mathcal{C}_-^{\ell+2}$ coincide with a section of the local stable manifold W_{loc}^s of T at $\bar{x}_{\ell+1}$. We claim that $\mathcal{C}_+^\ell = \mathcal{C}_-^{\ell+2}$. To prove this, assume the contrary statement. Then there exist points $x \in \mathcal{C}_+^\ell$ and $y \in \mathcal{C}_-^{\ell+2}$ such that $x \prec y$. Hence $T^n(x) \prec T^n(y)$ for all $n \geq 1$. Now for n large enough, both $T^n(x)$ and $T^n(y)$ enter W_{loc}^s , which is strongly ordered in the south-east order, so in particular $T^n(x)$ and $T^n(y)$ are not comparable in the north-east order. It follows that $\mathcal{C}_+^\ell = \mathcal{C}_-^{\ell+2}$. \square

Proof of Theorem 9 Assume both a and c are order stable from above. By Theorem 4 in [18] the boundary of the basin of attraction of a is a strongly south-east

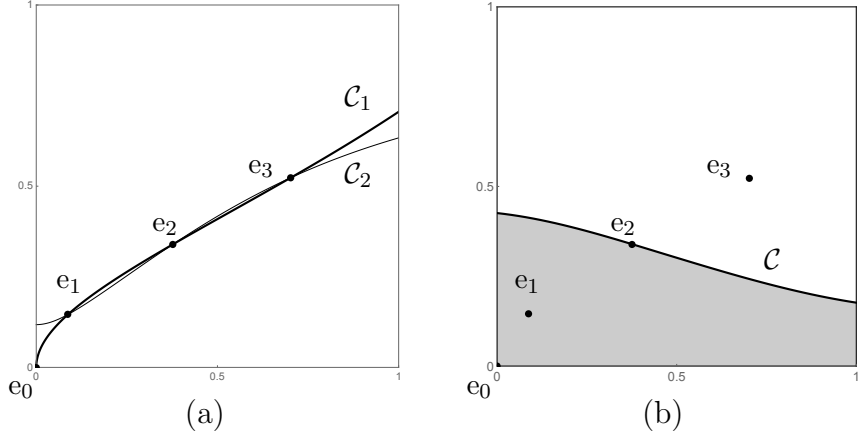


Figure 11. Case 3 in Table 1. There exist hyperbolic interior fixed points e_1 (stable), e_2 (unstable) and e_3 (stable). The origin e_0 is unstable. There are no minimal period-two points. (a) shows the equilibrium curves, which have a nontangential contact points at e_ℓ , $\ell = 1, 2, 3$. Either a calculation or Lemma 2 may be used to determine local stability of interior fixed points. (b) By Theorem 9 applied to the restriction of the map to the invariant order interval $[e_0, e_3]$, there exists a southeast ordered curve \mathcal{C} through e_e , such that non-zero points below \mathcal{C} are attracted to e_1 , points above \mathcal{C} are attracted to e_2 , and points on \mathcal{C} are attracted to e_2 . Since there are no period-two points outside $[e_0, e_3]$, the curve \mathcal{C} has an extension to a southeast ordered curve on \mathbb{R}_+^2 with endpoints on the boundary which separates the basins of attraction of e_1 and e_3 .

ordered curve \mathcal{C} . By the strong monotonicity of T and the Trichotomy Theorem, $c \in \mathcal{C}$, $[a, b] \setminus \{b\} \subset \mathcal{B}(a)$, and $[c, b] \setminus \{c\} \subset \mathcal{B}(b)$. The dynamics of the restriction of T to \mathcal{C} is one-dimensional on a compact interval with only one fixed point. Thus for every x in \mathcal{C} , $T^n(x)$ converges to c . Now for every y above \mathcal{C} with $y \neq b$, there exists x in \mathcal{C} such that $x \prec y$ and consequently $T^n(x) \prec T^n(y)$. Since $T^n(x) \rightarrow c$, then accumulation points z of $\{T^n(y)\}$ satisfy $c \preceq z$. Now if $z \neq c$, then $T(z)$ belongs to the interior of $[c, b]$. Thus $T^n(y)$ enters $[c, b]$ for some $n \in \mathbb{N}$, and therefore it converges to b . \square

Lemma 3 *Let a, b be fixed points of T with $a \prec b$ such that a and b are order stable with respect to $[a, b]$, and such that there exists a unique fixed point c of T satisfying $a \prec b \prec c$. Suppose $\mathcal{C} := \partial\mathcal{B}(a) \cap [a, b]$ has one and only one minimal period-two orbit $\{p, T(p)\}$, and \mathcal{C} has no minimal period-four points. If both smallest characteristic values of T^2 at c and at p are not ± 1 , then c is a repeller and p is*

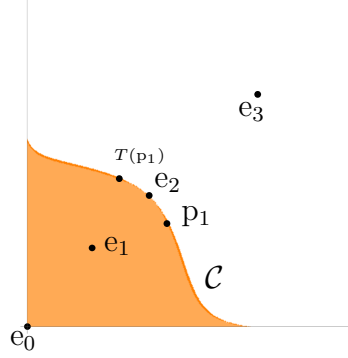


Figure 12. Case 4 in Table 1. T has hyperbolic fixed points e_0, e_1, e_2, e_3 , of which e_1 and e_3 are LAS, e_0 and e_2 are repellers. Also T has minimal period-two points $p_1, T(p_1)$, which are saddle points. Also, $\{p_1, T(p_1)\} \subset [e_1, e_3]$. By Theorem 10, there exists a south-east ordered curve \mathcal{C} that separates the basins of e_1 and e_3 in $[e_1, e_3]$ and which contains $p_1, e_2, T(p_1)$. By Remark 7 the curve \mathcal{C} has an extension with endpoints on the boundary of the nonnegative quadrant. Points on \mathcal{C} other than e_2 are attracted to the period-two orbit $\{p_1, T(p_1)\}$.

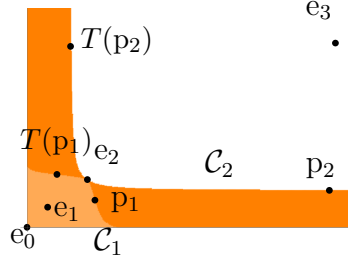


Figure 13. Case 5 in Table 1. Theorem 10 guarantees the existence of curves \mathcal{C}'_1 and \mathcal{C}'_2 in $[e_1, e_3]$ and through the period-two points and the point e_2 . These curves separate $[e_1, e_3]$ in regions attracted to e_1, e_3 , and $\{p_2, T(p_2)\}$. Theorem 4 and Corollary 2 in [18] implies that \mathcal{C}'_1 and \mathcal{C}'_2 can be extended to invariant curves \mathcal{C}_1 and \mathcal{C}_2 that bound the basin of attraction of e_1 and e_3 in \mathbb{R}^2_+ respectively. \mathcal{C}_1 and \mathcal{C}_2 are south-east ordered and extend to the boundary of \mathbb{R}^2_+ . The restriction of the map T^2 to each of the curves \mathcal{C}_1 and \mathcal{C}_2 exhibits one-dimensional dynamics of a bounded map on the real line that has two fixed points and no minimal period-two points. By Theorem C.3 in [8], iterates of points on \mathcal{C}_1 and \mathcal{C}_2 must converge to a fixed point of T^2 . The point e_2 is a repeller with only itself as pre-image. Consequently for $\ell = 1, 2$, for every point x in $\mathcal{C}_\ell \setminus \{e_2\}$ $T^n(x)$ is attracted to $\{p_\ell, T(p_\ell)\}$. In particular, iterates of points $x \in (\mathcal{C}_1 \cup \mathcal{C}_2) \setminus [e_1, e_3]$ must enter $[e_1, e_3]$ after a finite number of iterations. Since for every z in region between the curves there exist $x \in \mathcal{C}_1$ and $y \in \mathcal{C}_2$ such that $x \prec y \prec z$, Then $T^n(z)$ must enter $[e_1, e_3]$, and it is attracted to the nonhyperbolic minimal period two orbit. The curves \mathcal{C}_1 and \mathcal{C}_2 separate \mathbb{R}^2_+ into regions attracted to e_1, e_3 , and $\{p_2, T(p_2)\}$.

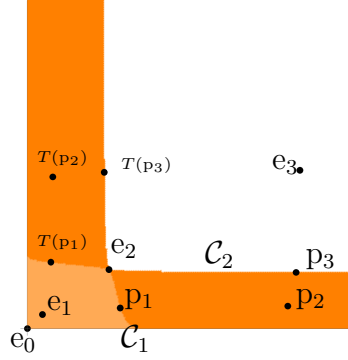


Figure 14. Case 6 in Table 1. T has hyperbolic fixed points e_0, e_1, e_2, e_3 , of which e_1 and e_3 are LAS, e_0 and e_2 are repellers. Also T has minimal period-two points $p_1, T(p_1), p_3, T(p_3)$ (saddle points), and $p_2, T(p_2)$ (LAS). Also, all minimal period-two points are in $[e_1, e_3]$. By Theorem 10, there exist south-east ordered curves \mathcal{C}_1 through $e_2, p_1, T(p_1)$ and \mathcal{C}_2 through $e_2, p_2, T(p_2)$. Iterates of points on \mathcal{C}_ℓ other than e_2 are attracted to the orbit $\{p_\ell, T(p_\ell)\}$, and points between the curves are attracted to $\{p_2, T(p_2)\}$. The curves \mathcal{C}_1 and \mathcal{C}_2 are part of the boundary of the basin of e_1 and e_3 respectively, thus by Remark 7 they have an extension to the boundary of the nonnegative quadrant.

a saddle point or a non-hyperbolic point of stable type. Furthermore, if no points in $\mathcal{C} \setminus \{c\}$ are mapped by T to c , then under iteration by T every point in $\mathcal{C} \setminus \{c\}$ is attracted to $\{p, T(p)\}$. An analogous statement holds true for the point b .

Proof. By Theorem 2 in [18], \mathcal{C} is a strongly ordered curve in the south-east ordering, with endpoints in $\partial[a, b]$. Due to strong monotonicity of T , the $\{p, T(p)\}$ is a subset of the interior of $[a, b]$. We need the following statement.

Claim: Suppose z is a fixed point or minimal period-two point of T in \mathcal{C} , and let τ and ρ be the characteristic values of T^2 at z , where $|\tau| < \rho$. If $\tau < 1$, then in every neighborhood \mathcal{V} of z there exist $x, y \in \mathcal{C} \cap \mathcal{V}$ such that $x \ll_{se} T^4(x) \leq_{se} z \leq_{se} T^4(y) \ll_{se} y$, and if $\tau > 1$, then in every neighborhood \mathcal{V} of z there exist $x, y \in \mathcal{C} \cap \mathcal{V}$ such that $T^4(x) \ll_{se} x \leq_{se} z \leq_{se} y \ll_{se} T^4(y)$. By Theorem 3 in [18], \mathcal{C} is tangential at z to the eigenspace associated to the characteristic value τ . The claim follows from this fact.

The set $\mathcal{C}' := \mathcal{C} \cap Q_4(c)$ is an unordered closed curve with c at one endpoint

Investigating existence of minimal period-four points through graphical means. Boundedness of the map implies that any minimal periodic point is in the order interval given by the smallest and largest fixed points. This determines the initial domain for a contour plot of $\|T^4(x) - x\|$. This first plot shows approximate locations of period four points, see the plot on the right. Then contour plots of $\|T^2(x) - x\|$ and $\|T^4(x) - x\|$ are produced with domains near these locations. If these plots show the same locations for values near zero, then this suggests that period-four points are actually period-two points, implying that there are no minimal period-four points.

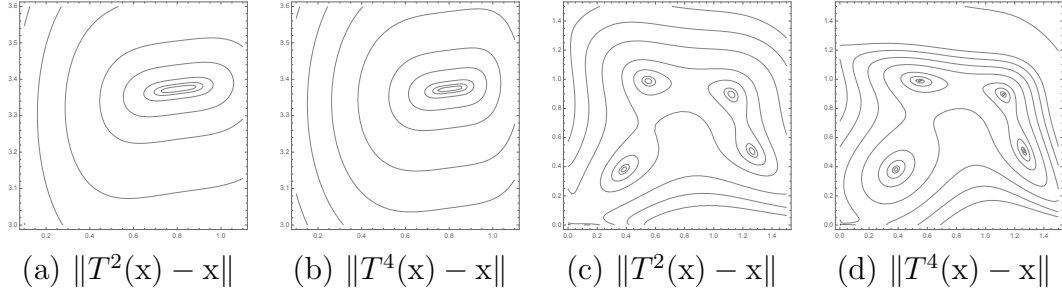
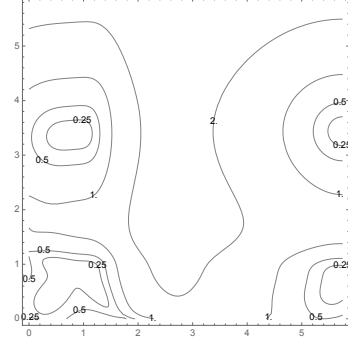


Figure 15. Zooming in on specific regions of the first contour plot shows more detail. (a) and (c) are contour plots of $\|T^2(x) - x\|$, and (b) and (d) are contour plots of $\|T^4(x) - x\|$. The plots suggest that the map has no minimal period-four points.

and with p (say) an interior fixed point. Denote with d the second endpoint of \mathcal{C}' . Note that \mathcal{C}' is invariant for T^2 , and that both c and p are the only fixed points of T^2 in \mathcal{C}' , and by hypothesis T^2 has no minimal period-two points in \mathcal{C}' .

We now introduce a parametrization of \mathcal{C}' . With $c = (c_1, c_2)$ and $d = (d_1, d_2)$, for $t \in [0, 1]$ define $\phi : [0, 1] \rightarrow \mathcal{C}'$ by $\phi(t) := (x, y)$ if $x = (1 - t)c_1 + td_1$ and y is such that $(x, y) \in \mathcal{C}'$. The function ϕ is well defined due to the strongly monotonic character of \mathcal{C}' . It is straightforward to verify that ϕ is one-to-one and onto, continuous, and satisfies $\phi(0) = c$, $\phi(1) = d$. Define $f : [0, 1] \rightarrow [0, 1]$ by $f(t) = \phi^{-1} \circ T^4 \circ \phi$, i.e., the following diagram commutes:

$$\begin{array}{ccc}
[0, 1] & \xrightarrow{f} & [0, 1] \\
\downarrow \phi & & \downarrow \phi \\
\mathcal{C}' & \xrightarrow{T^4} & \mathcal{C}'
\end{array}$$

Thus f has exactly two fixed points, namely 0 and a fixed point $t^* \in (0, 1)$ where $\phi(t^*) = p$. Note by the hypothesis on the point c the relation $f(t) = 0$ is only satisfied by $t = 0$. To see that p is not a repeller, assume it is. By the Claim above, the function $f(t)$ satisfies $f(t) > t$ for $t \in (t^*, d_1]$, which is not possible. Thus p is locally asymptotically stable or non-hyperbolic of stable type. In particular, t^* is locally asymptotically stable for $f(t)$. We now prove that c is a repeller. Assume the contrary, i.e., c is a saddle point. By the Claim and the fact that $f(t)$ has only two fixed points, it follows that $f(t) < t$ for $t \in [c_1, t^*)$. But this contradicts the Claim's conclusion of p being locally asymptotically stable. Thus c is a repeller.

Finally, by Theorem C.3 in [8] applied to $f(t)$ on $[0, 1]$, under iteration of f every point in $[0, 1]$ converges to 0 or t^* . But the basin of 0 consists of 0 only. Thus every point other than 0 converges under iteration to t^* , which implies the last statement in the Lemma. \square

The following result is a corollary to P. Hartman's Lemma 5.1 and Corollary 5.1 in [13].

Lemma 4 *Let $c \in \mathbb{R}^2$ be a fixed point of a planar map F which is of class C^1 in a neighborhood of c . Suppose that the characteristic values of F at c are real numbers τ and ρ such that $|\tau| < \min(1, \rho)$. Then there exists a C^1 curve \mathcal{C}^* through c that is locally invariant under F which is tangential to the eigenspace \mathcal{V} associated with τ , such that for any x , if $x \in \mathcal{C}^*$ then $T^n(x) \rightarrow c$, and if $x \notin \mathcal{C}^*$ and $F^n(x) \rightarrow c$ tangentially to \mathcal{V} , then there exists $n_0 \in \mathbb{N}$ such that $F^n(x) \in \mathcal{C}^*$ for $n \geq n_0$.*

Proof. There is no loss of generality in assuming $c = (0, 0)$ and that the map F

has the form

$$F(x, y) = (\tau x + f_1(x, y), \rho y + f_2(x, y)) \quad (44)$$

where f_1, f_2 and their first partial derivatives are all zero at $(0, 0)$. By Hartman's Lemma 5.1 there exists a function $y = \phi(x)$ of class C^1 for small $|x|$ satisfying $\phi(0) = \phi'(0) = 0$, and such that the graph of ϕ is locally invariant under F . By the same Lemma it may be assumed $\phi(x) = 0$ and $f_2(x, 0) = 0$ for small $|x|$, by performing a C^1 change of variables if necessary. The curve \mathcal{C}^* is now taken to consist of points $x = (x, 0)$ with small $|x|$. Choose a real number θ_0 so that $0 < \theta_0 < \min(\frac{\rho-\tau}{4}, \frac{1-\tau}{2})$. If $x = (x, 0) \in \mathcal{C}^*$, then $F(x, 0) = (\tau x + f_1(x, 0), 0)$, and by the proof of Corollary 5.1 in page 238 of [13], $|F(x, 0)| < (\tau + \theta_0)|x| < \frac{1+\tau}{2}|x|$. Hence $F^n(x) \rightarrow (0, 0)$. Now consider x such that $(x_n, y_n) := F^n(x)$ satisfies $x_n \neq 0$ for all $n \geq 0$, $(x_n, y_n) \rightarrow (0, 0)$ and $y_n/x_n \rightarrow 0$. To complete the proof it must be shown that $y_n = 0$ for all large enough n . If for some m the point (x_m, y_m) satisfies $y_m = 0$ and $|x_m|$ is small enough, then $y_{m+k} = 0$ for $k = 0, 1, 2, \dots$ and there is nothing else to prove. Now assume $y_n \neq 0$ for all $n \geq 0$. The proof of Corollary 5.1 in page 238 of [13] gives the inequality $|y_{n+1}| \geq (\rho - 2\theta_0)|y_n|$, which by the definition of θ_0 implies

$$|y_{n+1}| \geq \frac{1}{2}(\rho + \tau)|y_n|. \quad (45)$$

Since $f_1(x, y)$ and its derivatives are zero at the origin, we have

$$\frac{f_1(x_n, y_n)}{x_n} = \frac{o(|x_n| + |y_n|)}{x_n} = o\left(1 + \left|\frac{y_n}{x_n}\right|\right). \quad (46)$$

The assumption $y_n/x_n \rightarrow 0$ and (46) imply that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{f_1(x_n, y_n)}{x_n} \leq \frac{1}{2}(\rho - \tau), \quad n = n_0, n_0 + 1, \dots \quad (47)$$

From (44) and (47),

$$\begin{aligned} |x_{n+1}| &= |\tau x_n + f_1(x_n, y_n)| \leq (\tau + |f_1(x_n, y_n)/x_n|) |x_n|, \quad n = n_0, n_0 + 1, \dots \\ &\leq (\tau + \tfrac{1}{2}(\rho - \tau)) |x_n| = \tfrac{1}{2}(\rho + \tau) |x_n| \end{aligned} \quad (48)$$

Combine (45) and (48) to obtain

$$\left| \frac{y_{n+1}}{x_{n+1}} \right| \geq \left| \frac{y_n}{x_n} \right|, \quad n = n_0, n_0 + 1, \dots, \quad (49)$$

which contradicts the assumptions $y_n \neq 0$ and $y_n/x_n \rightarrow 0$. Thus $(x_n, y_n) \in \mathcal{C}^*$ for all n large enough. \square

Proof of Theorem 10. Lemma 3 and Theorem 3 in [18] imply most of statement (i), the only thing left to verify is that $\partial\mathcal{B}(a) = \partial\mathcal{B}(b)$. By Lemma 4 there is an invariant local curve \mathcal{C}^* through p that is tangential to the eigenvector associated with the smallest characteristic value of T at p . Since $\{p, p'\} \subset \partial\mathcal{B}(a) \cap \partial\mathcal{B}(b)$, we have $\mathcal{C}^* \subset \partial\mathcal{B}(a) \cap \partial\mathcal{B}(b)$. Arguing by contradiction, suppose $\partial\mathcal{B}(a) \neq \partial\mathcal{B}(b)$. Then there exist $x \in \mathcal{C}_1$, $y \in \mathcal{C}_2$ such that $x <_{ne} y$ neither point is a fixed point or a period-two point. Iterates $T^n(x)$ and $T^n(y)$ eventually enter \mathcal{C}^* by Lemma 4, so either they are non-comparable, or they are the same point. But either possibility is not allowed by the strongly cooperative character of the map T . Thus $\partial\mathcal{B}(a) = \partial\mathcal{B}(b)$.

To prove (ii), consider p_1 and p_2 as fixed points of the strongly cooperative map T^2 . Set $\mathcal{C}'_\ell := \mathcal{C}_\ell \cap \mathcal{Q}_4(C)$, $\ell = 1, 2$. Assume p_1 is a saddle point.

Since $p_1 \preceq p_2$, The Trichotomy Theorem [14] and p_1 being a saddle imply that $[p_1, p_2] \setminus \{p_1\} \subset \mathcal{B}(p_2)$. By Lemma 3, $\mathcal{C}'_\ell \setminus \{C\} \subset \mathcal{B}(p_\ell)$, $\ell = 1, 2$. If x is a point in the (interior of) region between \mathcal{C}'_1 and \mathcal{C}'_2 , there exist points $y_1 \in \mathcal{C}'_1$ and $y_2 \in \mathcal{C}'_2$ such that $y_1 \preceq x \preceq y_2$. Therefore $T^{2n}(y_1) \preceq T^{2n}(x) \preceq T^{2n}(y_2)$ for

$n = 1, 2, \dots$. Since $T^{2n}(y_\ell) \rightarrow p_\ell$ for $\ell = 1, 2$, for n large enough $T^{2n}(x) \in [p_1, p_2]$, and consequently $T^{2n}(x) \rightarrow p_2$ and $T^n(x) \rightarrow p_2$ as well. This completes the proof of (ii).

To prove (iii), note that for T^2 , the points p_1 and p_3 are saddle points and consequently p_2 is locally asymptotically stable. Note that $[p_1, p_3] \setminus \{p_1, p_3\} \subset \mathcal{B}(p_2)$. The argument used in the proof of part (ii) can be used here to conclude that the region between \mathcal{C}'_1 and \mathcal{C}'_2 is precisely the basin of p_2 as a fixed point of T^2 . This completes the proof of the theorem. \square

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MANUSCRIPT 4

**Cones for Coordinate-wise Monotone Functions and Dynamics of
Monotone Maps**

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Abstract

In this paper we give a characterization of monotone discrete systems of equations in terms of an associated matrix and give some properties of certain invariant surfaces of codimension 1, which often give the boundary of the basin of attraction of certain fixed points. We present several examples that illustrate our results in the case of k dimensional systems where $k \geq 3$.

4.1 Introduction and Preliminaries

In this paper we consider the maps on \mathbb{R}^n which are coordinate-wise monotonic and we characterize those maps which are monotone to a standard ordering \leq_σ of \mathbb{R}^n . In two dimensional case the obtained characterization will coincide with two classes of maps known as competitive and cooperative for which there is an extensive theory developed in [9, 10, 11, 17, 18, 19, 20, 21, 27]. A map $T = (f(x, y), g(x, y))$ is called competitive if $f(x, y)$ is non-decreasing in x and non-increasing in y and $g(x, y)$ is non-increasing in x and non-decreasing in y and it is called cooperative if both functions f, g are non-decreasing in x and y . This fact can be illustrated by using the signature matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad (50)$$

for cooperative and competitive case respectively. Here 1(resp. -1) means that the corresponding function is non-decreasing (resp. non-increasing) in its argument. As it was shown in sequence of papers [17, 18, 19, 20, 21, 27] cooperative and competitive maps in the plane have a lot of structure which leads to characterization of the *global* stable and unstable manifolds of all hyperbolic fixed and periodic points which in turn gives the boundaries of the basins of attraction of such points. In addition the union of global unstable manifolds gives the carrying

simplex that majority of solutions follow to the attracting fixed or periodic points. We were also able to obtain similar results for some non-hyperbolic fixed and periodic points, see [19, 20]. In this paper we will try to extend some of the results in [17, 18, 19, 20] to the case of maps in \mathbb{R}^n , $n \geq 3$. The major difficulty in this process is description of the boundary of an invariant surface which plays the role of global stable manifold in two dimensional case. While in two dimensional case the global stable manifolds are continuous curves their boundaries are always either fixed or periodic points or points on the boundary in higher dimensional case the global stable manifolds are surfaces which boundaries could have complicated structure, that makes the problem of their description harder. In this paper we will first characterize the maps monotone with respect to a standard ordering in \mathbb{R}^n among all coordinate-wise monotonic maps and then we will obtain the description of a certain invariant manifold, which plays the role of global stable manifold. We will illustrate our results with several examples. The next example demonstrate use of techniques from [17, 18, 19, 20] in proving global dynamics of a two-dimensional competitive map:

Example 3 Consider the system of difference equations

$$x_{n+1} = \frac{b_1 x_n}{1 + x_n + c_1 y_n}, \quad y_{n+1} = \frac{b_2 y_n}{1 + c_2 x_n + y_n} \quad n = 0, 1, \dots \quad (51)$$

where the parameters b_1, b_2, c_1 , and c_2 are positive real numbers and the initial conditions x_0 and y_0 are arbitrary non-negative numbers.

In a modelling setting, system (51) is well-known Leslie-Gower system [3] and is one of discrete version of Lotka-Volterra system of differential equations [16]. The state variables x_n and y_n denote population sizes during the n -th generation, and the sequence $\{(x_n, y_n) : n = 0, 1, 2, \dots\}$ depicts how the populations evolve over time. Competition between the two populations is reflected by the fact that

the transition function for each population is a decreasing function of the other population size.

Global behavior of (51) was considered in [3], and global behavior of related systems was considered in [1, 2, 6, 7, 18] and [19].

It has been shown in [3, 18] that under hypotheses

$$b_1 > 1, \quad b_2 > 1 \tag{52}$$

and

$$b_1 > 1 + c_1(b_2 - 1), \quad b_2 > 1 + c_2(b_1 - 1), \tag{53}$$

equation (51) has four equilibrium points: $E_0(0, 0)$, $E_1(b_1 - 1, 0)$, $E_2(0, b_2 - 1)$, and

$$E_3 \left(\frac{b_2 - 1}{c_1 c_2 - 1} \left(c_1 - \frac{b_1 - 1}{b_2 - 1} \right), \frac{b_1 - 1}{c_1 c_2 - 1} \left(c_2 - \frac{b_2 - 1}{b_1 - 1} \right) \right).$$

Theorem 4 in [3] states that E_1 and E_2 are saddle points, E_3 is locally asymptotically stable, and $E_0(0, 0)$ is a repeller. The same theorem states that E_3 is globally asymptotically stable, but the proof of this statement given in [3] is incomplete and it was completed in [19].

Theorem 11 *Consider system (51) subject conditions (52) and (53). Then every solution which starts in the interior of the positive quadrant converges to the positive equilibrium E_3 .*

Proof. We begin the proof by showing that the stable manifolds of E_1 and E_2 do not intersect the interior of the positive quadrant; more precisely,

$$\{(x, 0) : x \geq 0\} = W^s(E_1), \quad \{(0, y) : y \geq 0\} = W^s(E_2). \tag{54}$$

Clearly, if $x_0 = 0$ then $x_n = 0$ for every $n = 1, 2, \dots$ and if $y_0 = 0$ then $y_n = 0$ for

every $n = 1, 2, \dots$. Thus the global stable manifolds of the equilibrium points E_1 and E_2 satisfy:

$$\{x : x \geq 0\} \subset W^s(E_1), \quad \{y : y \geq 0\} \subset W^s(E_2).$$

In view of the uniqueness of the global stable manifold which follows from the Stable Manifold Theorem and the Hartman-Grobman Theorem, see [5], we obtain (54).

It is clear from (51) that $x_{n+1} \leq b_1$ and $y_{n+1} \leq b_2$ for $n = 0, 1, 2, \dots$. Therefore, without loss of generality we may drop the first term of the sequence if necessary, and assume (x_0, y_0) belongs to the rectangle $D := (0, b_1] \times (0, b_2]$. We shall need the sets $D_1 := (0, b_1^*) \times (0, b_2^*)$, $D_2 := D \setminus D_1$, and $L := \{(x, y) \in D : 1 + x + c_1 y = b_1 \text{ or } 1 + y + c_2 x = b_2\}$, where $b_1^* = \frac{b_2 - 1}{c_2}$ and $b_2^* = \frac{b_1 - 1}{c_1}$.

Denote by $Q_\ell(E_3)$, $\ell = 1, 2, 3, 4$ the four open sets consisting of the components of the open first quadrant minus the set L formed by the critical lines $1 + x + c_1 y = b_1$ and $1 + y + c_2 x = b_2$. The set D may be partitioned as the following disjoint union of sets:

$$D = (Q_1(E_3) \cap D_1) \cup (Q_1(E_3) \cap D_2) \cup Q_2(E_3) \cup Q_3(E_3) \cup Q_4(E_3) \quad (55)$$

Next we consider five cases labeled (a)–(e), corresponding to (x_0, y_0) being a member of one of the sets appearing in the right-hand-side of (55).

- (a) Take $(x_0, y_0) \in Q_1(E_3)$. Then $x_1 \leq x_0$, $y_1 \leq y_0$. Indeed $(x_0, y_0) \in Q_1(E_3)$ is equivalent with $1 + x_0 + c_1 y_0 > b_1$, $1 + c_2 x_0 + y_0 > b_2$ and it is equivalent to $x_1 \leq x_0$, $y_1 \leq y_0$.

Take $(x_0, y_0) \in Q_1(E_3) \cap D_1$. Then there exist points $(x_0^-, y_0^-) \in Q_2(E_3)$ and

$(x_0^+, y_0^+) \in Q_4(E_3)$ such that $(x_0^-, y_0^-) \leq (x_0, y_0) \leq (x_0^+, y_0^+)$, which implies $T^n((x_0^-, y_0^-)) \leq T^n((x_0, y_0)) \leq T^n((x_0^+, y_0^+))$ for all $n \geq 0$. Thus we obtain that

$$T^n((x_0, y_0)) \rightarrow \infty, \quad n \rightarrow \infty.$$

(b) Likewise, we obtain the analogue result if $(x_0, y_0) \in Q_3(E_3)$.

(c) Take $(x_0, y_0) \in Q_2(E_3)$. Then $1 + x_0 + c_1 y_0 < b_1, 1 + c_2 x_0 + y_0 > b_2$ which implies $x_1 > x_0, y_1 < y_0$ and so $(x_1, y_1) = T((x_0, y_0)) > (x_0, y_0)$. Using monotonicity of T we obtain that $\{(x_n, y_n)\}$ is an increasing sequence and so is convergent. The only limiting point to which it can converge is E_3 and thus

$$\lim_{n \rightarrow \infty} (x_n, y_n) = E_3.$$

Same conclusion holds if the initial point belongs to one of the critical lines.

(d) Take $(x_0, y_0) \in Q_4(E_3)$. Then $1 + x_0 + c_1 y_0 > b_1, 1 + c_2 x_0 + y_0 < b_2$ which implies $x_1 < x_0, y_1 > y_0$ and so $(x_1, y_1) = T((x_0, y_0)) < (x_0, y_0)$. Using monotonicity of T we obtain that $\{(x_n, y_n)\}$ is a decreasing sequence and so is convergent. The only limiting point to which it can converge is E_3 and thus

$$\lim_{n \rightarrow \infty} (x_n, y_n) = E_3.$$

Same conclusion holds if the initial point belongs to one of the critical lines.

(e) Next, take $(x_0, y_0) \in Q_1(E_3) \cap D_2$. Then $(x_1, y_1) = T((x_0, y_0)) \leq (x_0, y_0)$ that is $x_1 \leq x_0, y_1 \leq y_0$ because otherwise $(x_1, y_1) = T((x_0, y_0)) \geq (x_0, y_0)$ which would imply that $\{x_n, y_n\}$ is monotonic sequence in D_2 and so is convergent, which is a contradiction. Thus, $\{x_n, y_n\}$ is cw-monotonic sequence and so it must eventually enter the region D_1 . Otherwise, this sequence stays in D_2

and so is convergent which is impossible.

□

Remark 8 An alternative proof of this result can be obtained by showing that the associated map $T(x, y) = \left(\frac{b_1 x}{1+x+c_1 y}, \frac{b_2 y}{1+c_2 x+y} \right)$ satisfies so-called $(O+)$ condition [19, 27], which implies that every bounded orbit converges to a fixed point. However if condition

$$b_1 < 1 + c_1(b_2 - 1), \quad b_2 < 1 + c_2(b_1 - 1), \quad (56)$$

holds, then the fixed points E_1, E_2, E_3 exchange their local stability characters and so this changes global dynamics. In this case the existence and uniqueness of global stable manifold is essential, as the following result demonstrate, [14, 18, 19].

Theorem 12 *Consider system (51). Suppose that 56 holds, then E_1 and E_2 are globally asymptotically stable on their basins of attraction determined by the global stable manifold $W^s(E_3)$ which is the graph of a continuous, increasing function of the first coordinate. A solution $\{\mathbf{x}_n\}$ converges to E_1 whenever \mathbf{x}_0 is below $W^s(E_3)$ in South-east ordering, and $\{\mathbf{x}_n\}$ converges to E_2 whenever \mathbf{x}_0 is above $W^s(E_3)$ in South-east ordering.*

Let A be a subset of \mathbb{R}^n , and let $T : A \rightarrow A$ be a continuous function. We denote with T_ℓ the ℓ -th coordinate function of T , that is, $T = (T_1, \dots, T_n)$. We say that T is *or cw-monotonic* on A , if for $1 \leq i, j \leq n$, $T_i(x_1, \dots, x_n)$ is monotonic in x_j .

Let T be coordinate-wise monotonic map on $A \subset \mathbb{R}^n$ that is not constant on any coordinate. Define the *signature matrix* of T as the $n \times n$ matrix $M_T = \{m_{ij}\}$

with entries

$$m_{ij} = \begin{cases} 1 & \text{if } T_i(x_1, \dots, x_n) \text{ is nondecreasing in } x_j \\ -1 & \text{if } T_i(x_1, \dots, x_n) \text{ is nonincreasing in } x_j \end{cases}$$

Thus for cw-monotonic differentiable maps T with nonzero partial derivatives,

$$m_{ij} = \text{sign} \left(\frac{\partial T_i}{\partial x_j} \right).$$

Definition 7 For each choice of $\sigma = (\sigma_1, \dots, \sigma_n)$ with $\sigma_\ell \in \{-1, 1\}$, the standard cone associated to σ is the set

$$C_\sigma = \{(x_1, \dots, x_n) : \sigma_\ell x_\ell \geq 0 \text{ for } 1 \leq \ell \leq n\},$$

and the standard order associated to σ is the relation given by

$$x \leq_\sigma y \iff y - x \in C_\sigma$$

It is clear that there are 2^n distinct standard cones in \mathbb{R}^n . A map T on a set $A \subset \mathbb{R}^n$ is said to be *monotone with respect to the partial ordering \leq_σ* if $x \leq y \Rightarrow T(x) \leq_\sigma T(y)$ for $x, y \in A$.

4.2 Main Results

Theorem 13 Let A be a subset of \mathbb{R}^n with nonempty interior, and let $T : A \rightarrow A$ be a cw-monotonic map. A necessary and sufficient condition for T to be monotone with respect to a standard ordering \leq_σ of \mathbb{R}^n is that its signature matrix M_T has the form $M_T = \sigma^t \sigma$.

Proof. Suppose T is monotone increasing with respect to \leq_σ ; we wish to

prove $m_{ij} = \sigma_i \sigma_j$ for $i, j = 1, \dots, n$. Now consider x in the interior of A , and let $\delta \geq 0$ be small enough so that the closed ball centered at x with radius δ is contained in A . For i, j fixed in $\{1, \dots, n\}$ set $y = x + \delta e_j$, where e_j is coordinate vector which j -th component is 1. Hence $y_\ell - x_\ell = 0$ for $\ell \neq j$, and $y_j - x_j = \delta$. We now proceed to analyze the following two cases: $\sigma_j = 1$ and $\sigma_j = -1$. If $\sigma_j = 1$, then $x \leq_\sigma y$, which implies that $T(x) \leq_\sigma T(y)$. In particular, $\sigma_i(T_i(y) - T_i(x)) \geq 0$. If $\sigma_i = 1$, then $T_i(y) - T_i(x) \geq 0$, that is, T_i is nondecreasing in the j -th coordinate, so $m_{ij} = 1 = \sigma_i \sigma_j$. If $\sigma_i = -1$, then $T_i(x) - T_i(y) \geq 0$, that is, T_i is nonincreasing in the j -th coordinate, so $m_{ij} = -1 = \sigma_i \sigma_j$. In the second case $\sigma_j = -1$ we have $y \leq_\sigma x$, hence $T(y) \leq_\sigma T(x)$, and in particular, $\sigma_i(T_i(x) - T_i(y)) \geq 0$. If $\sigma_i = 1$, then $T_i(x) \geq T_i(y)$, that is, T_i is nonincreasing in the j -th coordinate, thus giving $m_{ij} = -1 = \sigma_i \sigma_j$. If $\sigma_i = -1$, then $T_i(y) \geq T_i(x)$, that is, T_i is nondecreasing in the j -th coordinate, thus giving $m_{ij} = 1 = \sigma_i \sigma_j$.

To prove sufficiency, suppose $M_T = \sigma^t \sigma$, and let x, y be such that $x \leq_\sigma y$. Then $\sigma_\ell(y_\ell - x_\ell) \geq 0$, for $\ell = 1, \dots, n$. That is,

$$x_\ell \leq y_\ell \text{ whenever } \sigma_\ell = 1, \text{ and } x_\ell \geq y_\ell \text{ whenever } \sigma_\ell = -1. \quad (57)$$

Suppose first that $\sigma_\ell = 1$. Then the ℓ -th row of M_T is equal to σ . This implies that T_ℓ is nondecreasing (respectively, nonincreasing) at the i -th coordinate if and only if $\sigma_i = 1$ (resp. $\sigma_i = -1$). In view of (57), we have that $T_\ell(y) - T_\ell(x) = \sigma_\ell(T_\ell(y) - T_\ell(x)) \geq 0$. If now $\sigma_\ell = -1$, then the ℓ -th row of M_T is equal to $-\sigma$. This implies that T_ℓ is nonincreasing (respectively, nondecreasing) at the i -th coordinate if and only if $\sigma_i = 1$ (resp. $\sigma_i = -1$). In view of (57), we have that $T_\ell(y) - T_\ell(x) = \sigma_\ell(T_\ell(y) - T_\ell(x)) \geq 0$. In all cases we get $T(x) \leq_\sigma T(y)$, i.e., T is monotone-increasing. \square

Theorem 14 *Let A be a bounded subset of \mathbb{R}^n and let $T : A \rightarrow A$ be a cw-monotone map. If $\mathbf{x} \in A$ is a subsolution or a supersolution, then $\{T^n(\mathbf{x})\}_{n \geq 0}$ converges in \mathbb{R}^n .*

Proof. The proof is similar to the proof of Theorem 13. If we take an arbitrary $\mathbf{x}_0 \in A_{\leq}$, then either $\mathbf{x}_1 = T(\mathbf{x}_0) \leq \mathbf{x}_0$. Using the monotonicity of T , we obtain a nonincreasing sequence $\{\mathbf{x}_n\}$. Thus all coordinates of $\{\mathbf{x}_n\}$ are monotonic bounded sequences and by monotone convergence principle are convergent. This implies that $\{\mathbf{x}_n\}$ is coordinate-wise convergent to $\bar{\mathbf{x}}$ and by using the continuity of T we see that $\bar{\mathbf{x}}$ is a fixed point of T . Similar reasoning can be applied in the case when $\mathbf{x}_0 \in A_{>}$. \square

Corollary 6 *When $n = 2, 3$ or 4 , a necessary and sufficient condition for the cw-monotonic map T to be monotonically increasing with respect to a standard ordering is that one of the following cases holds:*

(a) $n = 2$, and M_T is equal to one of the following matrices:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad (58)$$

(b) $n = 3$, and M_T is equal to one of the following matrices:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad (59)$$

(c) $n = 4$, and M_T is equal to one of the following matrices:

$$\begin{aligned}
& \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}, \\
& \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \\
& \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}. \tag{60}
\end{aligned}$$

Remark 1 For $n = 2$ the signature matrices coincide with the competitive and cooperative cases in the sense of M. Hirsch [27]. In the case $n = 3$ one of the signature matrices coincide with the cooperative case in the sense of M. Hirsch. Second and fourth signature matrix could be described as the matrices that describe the competition between two groups: one group consisting of two species and one group consisting of third species. In the case $n = 4$ one of the signature matrices coincide with the cooperative case in the sense of M. Hirsch. Second, fourth, and eighth signature matrix could be described as the matrices that describe the competition between two groups: one group consisting of three species and one group consisting of fourth species (second and eighth signature matrix) and two groups consisting of two species each (fourth signature matrix).

We now turn our attention to the properties of certain invariant surfaces of codimension 1 in \mathbb{R}^n . Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ be the extended real numbers. Then a partial order \preceq_σ on \mathbb{R}^n with positive cone given by a generalized octant extends in a natural way to a partial order on $\bar{\mathbb{R}}^n$, which we shall denote with the same symbol \preceq_σ .

For $z = (z_1, \dots, z_{n-1}, z_n)$ in $\bar{\mathbb{R}}^n$ with $n \geq 2$, we denote with $z^\pi = (z_1, \dots, z_{n-1})$

the projection of z onto $\bar{\mathbb{R}}^{n-1}$, and for $S \subset \bar{\mathbb{R}}^n$, set $S^\pi := \{z^\pi : z \in S\}$.

Theorem 15 *Let $p, q \in \bar{\mathbb{R}}^n$ be such that $p <_\sigma q$. With $R := \text{int}[p, q]_\sigma$, let $T : R \rightarrow R$ be a strongly monotone map. Set $B(p) := \{x \in R : T^m(x) \rightarrow p\}$. Define the function $\phi : R^\pi \rightarrow [p_n, q_n]$ as follows: set*

$$\phi(x_1, \dots, x_{n-1}) := q_n \quad \text{if } (x_1, \dots, x_{n-1}, q_n) \notin B(p) \quad (61)$$

and

$$\phi(x_1, \dots, x_{n-1}) := \sup \{t \in [p_n, q_n] : T^m(x_1, \dots, x_{n-1}, t) \rightarrow p\} \quad \text{if } (x_1, \dots, x_{n-1}, q_n) \in B(p) \quad (62)$$

If $\emptyset \neq B(p) \neq R$, then

- i. The function ϕ is continuous.
- ii. $\text{graph}(\phi) \cap R = \partial B(p) \cap R$.
- iii. $\text{graph}(\phi) \cap R$ consists of non-comparable points.
- iv. If T is differentiable on R and such that the n -th column of $T'(z)$ has positive entries for $z \in R$, then ϕ is Lipschitz on $(\text{graph}(\phi) \cap R)^\pi$.

Proof. For $x, y \in \bar{\mathbb{R}}^n$ such that $x \neq y$, the line segment $L(x, y) := \{(1 - \alpha)x + \alpha y : \alpha \in [0, 1]\}$ is said to be *vertical* if $x^\pi = y^\pi$.

Claim 9 *The closure of $\text{graph}(\phi) \cap R$ does not contain any vertical line segments.*

Proof. Suppose $[s, t]$ is a vertical line segment in the closure of $\text{graph}(\phi) \cap R$. Every small neighborhood of either one of the points s or t contains points in $B(p)$ and in its complement $R \setminus B(p)$. Thus a similar statement is true for the

points $T(s)$ and $T(t)$. Strong monotonicity of T and $s \preceq_\sigma t$ imply $T(s) <_\sigma T(t)$ and furthermore, positive real numbers δ_1 and δ_2 exist such that $T(B(s; \delta_1)) <_\sigma T(B(t; \delta_2))$. This relation together with the fact that $T(B(t; \delta_2))$ contains points in $\mathcal{B}(p)$, imply $T(B(s; \delta_1)) \subset \mathcal{B}(p)$, and consequently, $B(s; \delta_1) \subset \mathcal{B}(p)$. But this contradicts $B(s; \delta_1)$ having elements not in $B(p)$. \square

Claim 10 $\text{graph}(\phi) \cap R \neq \emptyset$.

Proof. There exist points x and y in R such that $p_n \leq \phi(x^\pi) < \phi(y^\pi) \leq q_n$, by the hypothesis on $B(p)$. If $p_n < \phi(x^\pi)$ then $(x^\pi, \phi(x^\pi)) \in \text{graph}(\phi) \cap R$ and if $\phi(y^\pi) < q_n$, then $(y^\pi, \phi(y^\pi)) \in \text{graph}(\phi) \cap R$, and in either case there is nothing else to prove. In the case when $p_n = \phi(x^\pi)$ and $\phi(y^\pi) = q_n$, set $f(t) := \phi(((1-t)x + ty)^\pi)$ for $t \in [0, 1]$. If there exists $t \in [0, 1]$ such that $p_n < f(t) < q_n$, then $((1-t)x + ty)^\pi, \phi(((1-t)x + ty)^\pi) \in \text{graph}(\phi) \cap R$, so suppose there is no such t . Then there exist $t_* \in [0, 1]$ and sequences $\{t_\ell\}$ and $\{s_\ell\}$ in $[0, 1]$ such that $t_\ell \rightarrow t_*$, $f(t_\ell) \rightarrow p_n$ and $s_\ell \rightarrow t_*$, $f(s_\ell) \rightarrow q_n$. It follows that the vertical line segment with endpoints $((1-t_*)x + t_*y)^\pi, p_n$ and $((1-t_*)x + t_*y)^\pi, q_n$ is a subset of the closure of $\text{graph}(\phi) \cap R$, which contradicts Claim 9. \square

Claim 11 *The restriction of ϕ to $(\text{graph}(\phi) \cap R)^\pi$ is decreasing in all variables.*

Proof. Let $x \in \text{graph}(\phi) \cap R$ and e be an element of the standard basis of \mathbb{R}^n different from $(0, \dots, 0, 1)$. Note that e^π is an element of the standard basis of \mathbb{R}^{n-1} , and for $h \in \mathbb{R}$, $(x + h e)^\pi = x^\pi + h e^\pi$. Choose $h > 0$ small enough so that $x + h e \in R$ and consequently $x^\pi + h e^\pi \in R^\pi$. In this case, $x \preceq_\sigma x + h e$, and strict monotonicity of T implies $T(x) <_\sigma T(x + h e)$. The latter relation and the definition of ϕ give the relation $\phi((x + h e)^\pi) \leq \phi(x^\pi)$, that is, $\phi(x^\pi + h e^\pi) \leq \phi(x^\pi)$. Now strong

monotonicity of T and the argument used in Claim 9 imply $\phi(\mathbf{x}^\pi + h \mathbf{e}^\pi) < \phi(\mathbf{x}^\pi)$.

□

Proof of i. ϕ is continuous. Suppose that $\mathbf{x} \in R$ is such that ϕ is not continuous at \mathbf{x}^π . Then there exists a sequence $\{x_m\}$ in R with $\mathbf{x}_m^\pi \rightarrow \mathbf{x}^\pi$ and $\phi(\mathbf{x}_m^\pi) \rightarrow \beta$, where $\beta \neq \phi(\mathbf{x}^\pi)$. Therefore, the vertical line segment with endpoints $(\mathbf{x}^\pi, \phi(\mathbf{x}^\pi))$ and (\mathbf{x}^π, β) is a subset of the closure of $\text{graph}(\phi) \cap R$. This contradicts Claim 9.

Proof of ii. If $\mathbf{x} \in \text{graph}(\phi) \cap R$, then $p_n < \phi(\mathbf{x}) < q_n$, and by the definition of ϕ , any neighborhood of \mathbf{x} has common points with $B(\mathbf{p})$ and with $R \setminus B(\mathbf{p})$. That is, $\mathbf{x} \in \partial B(\mathbf{p}) \cap R$. The argument is reversible.

Proof of iii. Follows from Claim 11.

Proof of iv. For T as in the hypothesis of iv, assume $\mathbf{x}^* \in \text{graph}(\phi) \cap R$ is such that T is not Lipschitz at $(\mathbf{x}^*)^\pi$. Then there exists a sequence \mathbf{x}^m in $\text{graph}(\phi)$ such that $\mathbf{x}^m \rightarrow \mathbf{x}^*$, $(\mathbf{x}^m)^\pi \neq (\mathbf{x}^*)^\pi$ for $m \geq 1$ and

$$\frac{\|(\mathbf{x}^m)^\pi - (\mathbf{x}^*)^\pi\|}{|\phi((\mathbf{x}^m)^\pi) - \phi((\mathbf{x}^*)^\pi)|} = \frac{\|(\mathbf{x}^m)^\pi - (\mathbf{x}^*)^\pi\|}{|x_n^m - x_n^*|} \rightarrow 0 \quad \text{as } m \rightarrow +\infty \quad (63)$$

There is no loss of generality in assuming $x_n^m - x_n^* > 0$ for $m = 1, 2, \dots$. Let $\mathbf{e} = (0, \dots, 0, 1)$ be the n -th member of the standard basis of \mathbb{R}^n . We have,

$$\frac{\|T'_{x^*}(\mathbf{x}^m - \mathbf{x}^* - (x_n^m - x_n^*)\mathbf{e})\|}{x_n^m - x_n^*} = \frac{\|T'_{x^*}(x_1^m - x_1^*, \dots, x_{n-1}^m - x_{n-1}^*, 0)\|}{x_n^m - x_n^*} \leq \|T'_{x^*}\| \frac{\|(\mathbf{x}^m)^\pi - (\mathbf{x}^*)^\pi\|}{x_n^m - x_n^*} \quad (64)$$

Differentiability of T at \mathbf{x}^* gives that as $m \rightarrow +\infty$,

$$\frac{1}{x_n^m - x_n^*} (T(\mathbf{x}^m) - T(\mathbf{x}^*)) - T'_{x^*}(\mathbf{e}) = \frac{1}{x_n^m - x_n^*} (T'_{x^*}(\mathbf{x}^m - \mathbf{x}^* - (x_n^m - x_n^*)\mathbf{e}) + o(1)) \quad (65)$$

Putting together relations (63), (64) and (65) we obtain

$$\frac{1}{x_n^m - x_n^*} (T(x^m) - T(x^*)) = T'_{x^*}(e) + o(1) \quad \text{as } m \rightarrow +\infty. \quad (66)$$

Since $T'_{x^*}(e)$ is precisely the n -th column of T'_{x^*} , equation (46) implies that for m large enough, $T(x^*) <_\sigma T(x^m)$. This contradicts iii.

□

4.3 Examples

Example 4 Consider the following difference equation

$$x_{n+1} = x_n \phi_1(a_1 x_n - d_1), \quad n = 0, 1, \dots \quad (67)$$

where $a_1, d_1 > 0$, ϕ_1 is such that

$$\phi_1 : \mathbb{R}^m \rightarrow \mathbb{R}_+^m, \phi_1 \in C', \phi'_1 > 0, \phi_1 > 0, \phi_1(0) = 1. \quad (68)$$

The equilibrium equation of (67) is

$$x = x \phi_1(a_1 x - d_1)$$

which gives $E_0 = 0$ and $E_x = \frac{d_1}{a_1}$ as the two fixed points to (67).

Lemma 5 *Given (67), E_0 is locally asymptotically stable and E_x is repeller for all parameter values.*

Proof. First set $x_{n+1} = f(x_n)$ where $f(x) = x \phi_1(a_1 x - d_1)$, then

$$f'(x) = \phi_1(a_1 x - d_1) + a x \phi'_1(a_1 x - d_1).$$

Now $f'(0) = \phi_1(-d_1) \in (0, 1)$ so E_0 is locally asymptotically stable. Also $f'(\frac{d_1}{a_1}) = \phi_1(0) + d_1\phi_1'(0) > 1$, therefore, E_x is repeller. \square

For the global dynamics of (67) consider the sets of points $U = \{x : x < E_x\}$ and $V = \{x : x > E_x\}$. If $x \in U$, then $\lim_{n \rightarrow \infty} T^n(x) = 0$ since 0 is a locally asymptotically stable fixed point. Now consider $x \in V$. For a contradiction assume for $x \in V$, the solution is bounded. Then $\lim_{n \rightarrow \infty} x_n = x_*$. So, $\lim_{n \rightarrow \infty} T(x_n) = T(x_*)$, which implies, $\lim_{n \rightarrow \infty} x_{n+1} = T(x_*) = x_*$. Thus x_* is a fixed point of (67), a contradiction. Thus for $x \in V$, $\lim_{n \rightarrow \infty} T^n(x) = \infty$.

Example 5 Consider the following system of equations

$$\begin{aligned} x_{n+1} &= x_n \phi_1(a_1 x_n + b_1 y_n - d_1) \\ y_{n+1} &= y_n \phi_2(a_2 x_n + b_2 y_n - d_2) \end{aligned} \quad n = 0, 1, \dots \quad (69)$$

where $a_i, b_i, d_i > 0$ for $i = 1, 2$, $x_0, y_0 \geq 0$ and ϕ_1, ϕ_2 are as described in (68). Also, let $S : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ be the map associated to (69). The Jacobian of S is

$$J(S) = \begin{bmatrix} \phi_1(-d_1 + a_1 x + b_1 y) & b_1 x \phi_1'(-d_1 + a_1 x + b_1 y) \\ +a_1 x \phi_1'(-d_1 + a_1 x + b_1 y) & \\ a_2 y \phi_2'(-d_2 + a_2 x + b_2 y) & \phi_2(-d_2 + a_2 x + b_2 y) \\ & +b_2 y \phi_2'(-d_2 + a_2 x + b_2 y) \end{bmatrix}$$

Since all of the entries of $J(S)$ are positive, we see that (69) is a cooperative system.

The equilibrium equations of (69) are given by,

$$\begin{aligned} a_1 x + b_1 y - d_1 &= 0 \\ a_2 x + b_2 y - d_2 &= 0 \end{aligned} \quad (70)$$

The solutions of (70) give the following four equilibrium points, $E_0 = (0, 0)$, $E_x = (\frac{d_1}{a_1}, 0)$, $E_y = (0, \frac{d_2}{b_2})$ and $E_{xy} = (\frac{b_1 d_2 - b_2 d_1}{a_2 b_1 - a_1 b_2}, \frac{a_2 d_1 - a_1 d_2}{a_2 b_1 - a_1 b_2})$. Notice E_0, E_x and E_y will

always exist, however for E_{xy} to exist,

$$\frac{b_1d_2 - b_2d_1}{a_2b_1 - a_1b_2} > 0 \quad \text{and} \quad \frac{a_2d_1 - a_1d_2}{a_2b_1 - a_1b_2} > 0.$$

Notice for these inequalities to hold, the numerator and denominator of these fractions must be either both positive or both negative.

Furthermore, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{d_1}{d_2}$, then there are an infinite number of fixed points along the line $y = -\frac{a_1}{b_1}x + \frac{d_1}{b_1}$.

Lemma 6 *Given (69), E_0 is locally asymptotically stable, while E_x , E_y and E_{xy} are unstable for all values of parameters. Furthermore, if E_x and E_y are saddles, then E_{xy} is a source and if E_x and E_y are sources, then E_{xy} is a saddle.*

Proof. The roots of the characteristic polynomial given by $J(S(E_0))$ are $\lambda_1 = \phi_1(-d_1)$, $\lambda_2 = \phi_2(-d_2)$. Since $d_1, d_2 > 0$, it follows that $|\lambda_1|, |\lambda_2| < 1$, so E_0 is locally asymptotically stable.

Next for the stability character of the equilibrium points on the axes. Starting with E_x , the roots of the characteristic polynomial of $J(S(E_x))$ are $\lambda_1 = 1 + d_1$, $\lambda_2 = \phi_2(\frac{a_2d_1}{a_1} - d_2)$. Since $d_1 > 0$, $|\lambda_1| > 1$ always. So E_x is always unstable. Now if $a_2d_1 > a_1d_2$, then E_x is a source and if $a_2d_1 < a_1d_2$, then E_x is a saddle. By a similar argument for E_y , we get E_y is also always unstable. It is a source when $b_1d_2 > b_2d_1$ and a saddle when $b_1d_2 < b_2d_1$.

Lastly we will investigate the stability character of E_{xy} . Using the computer algebra system *Mathematia*, we can find the solutions to the characteristic polynomial of $J(S(E_{xy}))$, we will call them λ_1 and λ_2 . Now λ_1, λ_2 have different values depending on whether or not E_x, E_y are sources or saddles. If E_x, E_y are sources, then $\lambda_1 < 1$ while $\lambda_2 > 1$, thus E_{xy} is a saddle. If E_x, E_y are saddles, then $\lambda_1, \lambda_2 > 1$, thus E_{xy} is a source. Thus we have the condition that E_{xy} is a source if

$a_2d_1 < a_1d_2$ and $b_1d_2 < b_2d_1$. Also E_{xy} is a saddle if $a_2d_1 > a_1d_2$ and $b_1d_2 > b_2d_1$. Notice that it is not possible for E_y to be a saddle and E_x to be a source, or the other way around, else the condition for the existence of E_{xy} is violated. Thus we have shown that E_x , E_y and E_{xy} are always unstable, completing the proof.

□

Global dynamics of Equation (69) can be derived directly from Theorems 8-10 in [13].

Example 6 Consider the following system of equations

$$\begin{aligned} x_{n+1} &= x_n \phi_1(a_1x_n + b_1y_n + c_1z_n - d_1) \\ y_{n+1} &= y_n \phi_2(a_2x_n + b_2y_n + c_2z_n - d_2) \quad n = 0, 1, \dots \\ z_{n+1} &= z_n \phi_3(a_3x_n + b_3y_n + c_3z_n - d_3) \end{aligned} \quad (71)$$

where $a_i, b_i, c_i, d_i > 0$ for $i = 1, 2, 3$, $x_0, y_0, z_0 \geq 0$, and ϕ_1, ϕ_2, ϕ_3 are as described in (68). Also, let $T : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+^3$ be the map associated to (71). The Jacobian of T , $J(T)$ is shown below.

$$\begin{bmatrix} \phi_1(-d_1 + a_1x + b_1y + c_1z) & b_1x\phi_1'(-d_1 + a_1x + b_1y + c_1z) & c_1x\phi_1'(-d_1 + a_1x + b_1y + c_1z) \\ +a_1x\phi_1'(-d_1 + a_1x + b_1y + c_1z) & & \\ a_2y\phi_2'(-d_2 + a_2x + b_2y + c_2z) & \phi_2(-d_2 + a_2x + b_2y + c_2z) & c_2y\phi_2'(-d_2 + a_2x + b_2y + c_2z) \\ & +b_2y\phi_2'(-d_2 + a_2x + b_2y + c_2z) & \\ a_3z\phi_3'(-d_3 + a_3x + b_3y + c_3z) & b_3z\phi_3'(-d_3 + a_3x + b_3y + c_3z) & \phi_3(-d_3 + a_3x + b_3y + c_3z) \\ & & +c_3z\phi_3'(-d_3 + a_3x + b_3y + c_3z) \end{bmatrix}$$

Since all of the entries of $J(T)$ are positive, we see that (71) is a cooperative system.

The equilibrium equations of (71) are given by,

$$\begin{aligned} a_1x + b_1y + c_1z - d_1 &= 0 \\ a_2x + b_2y + c_2z - d_2 &= 0 \\ a_3x + b_3y + c_3z - d_3 &= 0 \end{aligned} \tag{72}$$

The solutions of (72) gives the following equilibrium points, $E_0 = (0, 0, 0)$,
 $E_x = (\frac{d_1}{a_1}, 0, 0)$, $E_y = (0, \frac{d_2}{b_2}, 0)$, $E_z = (0, 0, \frac{d_3}{c_3})$, $E_{xy} = (\frac{b_1d_2 - b_2d_1}{a_2b_1 - a_1b_2}, \frac{a_2d_1 - a_1d_2}{a_2b_1 - a_1b_2}, 0)$,
 $E_{xz} = (\frac{c_1d_3 - c_3d_1}{a_3c_1 - a_1c_3}, 0, \frac{a_3d_1 - a_1d_3}{a_3c_1 - a_1c_3})$, $E_{yz} = (0, \frac{c_2d_3 - c_3d_2}{b_3c_2 - b_2c_3}, \frac{b_3d_2 - b_2d_3}{b_3c_2 - b_2c_3})$, and $E_{xyz} = (\bar{x}, \bar{y}, \bar{z})$,
where

$$\begin{aligned} \bar{x} &= \frac{b_3c_2d_1 - b_2c_3d_1 - b_3c_1d_2 + b_1c_3d_2 + b_2c_1d_3 - b_1c_2d_3}{a_3b_2c_1 - a_2b_3c_1 - a_3b_1c_2 + a_1b_3c_2 + a_2b_1c_3 - a_1b_2c_3} \\ \bar{y} &= \frac{a_3c_2d_1 + a_2c_3d_1 + a_3c_1d_2 - a_1c_3d_2 - a_2c_1d_3 + a_1c_2d_3}{a_3b_2c_1 - a_2b_3c_1 - a_3b_1c_2 + a_1b_3c_2 + a_2b_1c_3 - a_1b_2c_3} \\ \bar{z} &= \frac{a_3b_2d_1 - a_2b_3d_1 - a_3b_1d_2 + a_1b_3d_2 + a_2b_1d_3 - a_1b_2d_3}{a_3b_2c_1 - a_2b_3c_1 - a_3b_1c_2 + a_1b_3c_2 + a_2b_1c_3 - a_1b_2c_3} \end{aligned}$$

Similar to system (69) E_0, E_x, E_y and E_z will always exist. While for E_{xy}, E_{xz}, E_{yz} and E_{xyz} to exist, each of the coordinates must be non-negative. Thus, the numerators and denominators of these fractions must be either both positive or both negative.

First notice that if we consider only the points on the x -axis, y -axis, or z -axis the results of Lemma 1 apply because if we restrict (71) to one variable, we have (67). Similarly if we consider points only in the xy -plane, xz -plane or yz -plane the results of lemma 2 apply because if we restrict (71) to two variables we have (69). So we only need consider points in the interior of the positive octant.

The global dynamics of (71) are described in the following theorem which follows from Theorem 15.

Theorem 16 *Let $p, q \in \mathbb{R}^3$ such that $p <_\sigma q$ where $\sigma = (1, 1, 1)$ and $R \subset \mathbb{R}^3$ such that*

$R = \text{int}[p, q]$, and let $T : R \rightarrow R$ be a strongly cooperative map. For $x =$

$(x_1, x_2, x_3) \in \mathbb{R}^3$, set $\mathcal{B}(p) = \{x \in R : T^m(x) \rightarrow p\}$ and define the function $\psi : R \rightarrow [[p_n, q_n]]$ as follows:

Set

$$\psi(x_1, x_2) = q_n \text{ if } (x_1, x_2, q_n) \notin \mathcal{B}(p)$$

and set

$$\psi(x_1, x_2) = \sup\{t \in [[p_n, q_n]] : T^m(x_1, x_2, t) \rightarrow p\} \text{ if } (x_1, x_2, q_n) \in \mathcal{B}(p)$$

If $\mathcal{B}(p) \neq R \neq \emptyset$ then ψ is a continuous function such that $\text{graph}(\psi) \cap R = \partial\mathcal{B}(p) \cap R$ where $\text{graph}(\psi) \cap R$ is a surface that contains all fixed points of T . Furthermore, all the points on $\text{graph}(\psi) \cap R$ are non-comparable with respect to σ . Here $p = (0, 0, 0)$.

Example 7 Consider the following cooperative system of equations:

$$\begin{aligned} x_{n+1} &= ax_n + b \frac{y_n}{1+y_n} \\ y_{n+1} &= c \frac{z_n}{1+z_n} + dy_n \\ z_{n+1} &= e \frac{x_n}{1+x_n} + fz_n \end{aligned} \tag{73}$$

where $a, b, c, d, e, f > 0$, $n = 0, 1, 2, \dots$ and $x_0, y_0, z_0 \geq 0$.

To begin we will discuss the existence of fixed points in the above system of equations. Fixed points $(\bar{x}, \bar{y}, \bar{z})$ of (73) satisfy:

$$\begin{aligned} (1-a)x &= b \frac{y}{1+y} \\ (1-d)x &= c \frac{z}{1+z} \\ (1-f)x &= e \frac{x}{1+x} \end{aligned} \tag{74}$$

Set

$$A = bce - (1-a)(1-d)(1-f) \tag{75}$$

This system always has the fixed point $E_0(0, 0, 0)$, and also has the fixed point

$E_1(\bar{x}, \bar{y}, \bar{z})$ when $A > 0$ and $a, d, f < 1$ where,

$$\begin{aligned}\bar{x} &= \frac{A}{(1-a)((1-d)(1-f)+e(1-d+c))} \\ \bar{y} &= \frac{A}{(1-d)((1-a)(1-f)+b(1-f+e))} \\ \bar{z} &= \frac{A}{(1-f)((1-a)(1-d)+c(1-a+b))}\end{aligned}\tag{76}$$

Theorem 17 *Every solution of (73) satisfies one of the following in the chart below.*

$\lim_{n \rightarrow \infty} (x_n, y_n, z_n)$	Conditions
(∞, ∞, ∞)	$a \geq 1, d \geq 1, f \geq 1$
$(\infty, \frac{e}{1-d}, \infty)$	$a \geq 1, d < 1, f \geq 1$
$(\infty, \infty, \frac{e}{1-f})$	$a \geq 1, d \geq 1, f < 1$
$(\frac{b}{1-a}, \infty, \infty)$	$a < 1, d \geq 1, f \geq 1$
$(\infty, \frac{ce}{(1-d)(1-f+e)}, \frac{e}{1-f})$	$a \geq 1, d < 1, f < 1$
$(\frac{bc}{(1-a)(1-d+c)}, \frac{c}{1-d}, \infty)$	$a < 1, d < 1, f \geq 1$
$(\frac{b}{1-a}, \infty, \frac{be}{(1-f)(1-a+b)})$	$a < 1, d \geq 1, f < 1$
$(0, 0, 0)$	$a < 1, d < 1, f < 1, A \leq 0$
$(\bar{x}, \bar{y}, \bar{z})$	$a < 1, d < 1, f < 1, A > 0$

Proof: Consider x_n and assume $a \geq 1, d \geq 1, f \geq 1$. Then it follows immediately that $\lim_{n \rightarrow \infty} x_n = \infty$, we get the same result for y_n and z_n as they have the same form, thus $\lim_{n \rightarrow \infty} (x_n, y_n, z_n) = (\infty, \infty, \infty)$.

Now assume $a \geq 1, d \geq 1, f < 1$. Then $z_{n+1} = e \frac{x_n}{1+x_n} + fz_n = e + fz_n$ since $\lim_{n \rightarrow \infty} x_n = \infty$. Now $z_{n+1} = e + fz_n$ is linear, so we have $\lim_{n \rightarrow \infty} z_n = \frac{e}{1-f}$. Thus $\lim_{n \rightarrow \infty} (x_n, y_n, z_n) = (\infty, \infty, \frac{e}{1-f})$. A similar proof gives when $a \geq 1, d < 1, f \geq 1$, $\lim_{n \rightarrow \infty} (x_n, y_n, z_n) = (\infty, \frac{c}{1-d}, \infty)$ and when $a < 1, d \geq 1, f \geq 1$, $\lim_{n \rightarrow \infty} (x_n, y_n, z_n) = (\frac{b}{1-a}, \infty, \infty)$.

Now consider when $a \geq 1, d < 1, f < 1$. Since $f < 1$, from the above $\lim_{n \rightarrow \infty} z_n = \frac{e}{1-f}$. Then $y_{n+1} = c \frac{z_n}{1+z_n} + dy_n = c \frac{\frac{e}{1-f}}{1+\frac{e}{1-f}} + dy_n = \frac{ce}{1-f+e} + dy_n$. So $\lim_{n \rightarrow \infty} y_n = \frac{\frac{ce}{1-f+e}}{1-d} = \frac{ce}{(1-d)(1-f+e)}$. So we get $\lim_{n \rightarrow \infty} (x_n, y_n, z_n) =$

$(\infty, \frac{ce}{(1-d)(1-f+e)}, \frac{e}{1-f})$. A similar proof gives when $a < 1, d \geq 1, f < 1$, $\lim_{n \rightarrow \infty} (x_n, y_n, z_n) = (\frac{b}{1-a}, \infty, \frac{be}{(1-f)(1-a+b)})$ and when $a < 1, d < 1, f \geq 1$, $\lim_{n \rightarrow \infty} (x_n, y_n, z_n) = (\frac{bc}{(1-a)(1-d+c)}, \frac{c}{1-d}, \infty)$.

Now consider when $a, d, f < 1$. There are two cases in this instance, $A > 0$ and $A \leq 0$. The characteristic polynomial for E_0 is

$$G(\lambda) := \lambda^3 - (a + d + f)\lambda^2 + (ad + df + fa)\lambda - (adf + bce) = 0$$

Let

$$F(\lambda) := (\lambda - a)(\lambda - d)(\lambda - f)$$

then,

$$G(\lambda) = F(\lambda) - bce$$

Consider the case when $A > 0$. $A > 0$ if and only if $bce > F(1)$ by definition of A and $F(\lambda)$. Also, $bce > F(1)$ if and only if $G(1) < 0$ by definition of $G(\lambda)$. So $G(1) < 0$, and $G(\infty) = \infty$, therefore there must exist a $\lambda \in (1, \infty)$ such that $G(\lambda) = 0$. Thus E_0 is a repeller, so by Theorem 4 in [4] E_1 must be an attractor inside the order interval $\llbracket E_0, E_1 \rrbracket = \{y : E_0 \leq y \leq E_1\}$. For points outside of $\llbracket E_0, E_1 \rrbracket$, we consider the following claim.

Claim: If (α, α, α) is our initial point with α large, we have the following inequalities:

$$\begin{aligned} \frac{b}{1-a} &< 1 + \alpha \\ \frac{c}{1-d} &< 1 + \alpha \\ \frac{e}{1-f} &< 1 + \alpha \end{aligned} \tag{77}$$

Proof of claim: If (α, α, α) is our initial point with α large, we have the following:

$$T(\alpha, \alpha, \alpha) = \left(a\alpha + b\frac{\alpha}{1+\alpha}, d\alpha + c\frac{\alpha}{1+\alpha}, f\alpha + e\frac{\alpha}{1+\alpha} \right)$$

Consider first $a\alpha + b\frac{\alpha}{1+\alpha} < \alpha$. Then $a + \frac{b}{1+\alpha} < 1$ if and only if $\frac{b}{1+\alpha} < 1 - a$ if and only if $\frac{b}{1-a} < 1 + \alpha$. For α large enough, this inequality holds. A similar calculation yields $d\alpha + c\frac{\alpha}{1+\alpha} < \alpha$ if and only if $\frac{c}{1-d} < 1 + \alpha$ and $f\alpha + e\frac{\alpha}{1+\alpha} < \alpha$ if and only if $\frac{e}{1-f} < 1 + \alpha$, proving the claim.

By the claim, for any initial point (x_0, y_0, z_0) outside of $\llbracket E_0, E_1 \rrbracket$, E_1 is an attracting point. Since E_1 is an attractor for points inside and outside of $\llbracket E_0, E_1 \rrbracket$, $\lim_{n \rightarrow \infty} (x_n, y_n, z_n) = (\bar{x}, \bar{y}, \bar{z})$.

Next the case when $A \leq 0$. By (73) there is only one equilibrium point $E_0(0, 0, 0)$. Now $A < 0$ if and only if $bce < F(1)$ by definition of A and $F(\lambda)$ and $bce < F(1)$ if and only if $G(1) > 0$ by definition of $G(\lambda)$. So $G(1) > 0$, therefore all roots of $G(\lambda)$ must be in the interval $(0, 1)$. Since all roots of the characteristic polynomial are less than 1, E_0 is a locally asymptotically stable in this case. Now $A = 0$ if and only if $G(1) = 0$, so in this case E_0 is non-hyperbolic, but we will show that E_0 is still an attractor in this instance. Consider $A \leq 0$, and the previous claim. Since E_0 is the only equilibrium point and the inequalities from the claim still hold, E_0 is an attractor. Thus $\lim_{n \rightarrow \infty} (x_n, y_n, z_n) = (0, 0, 0)$ for $A \leq 0$, completing the proof.

Example 8 Consider the following system of equations:

$$\begin{aligned} x_{n+1} &= x_n(1 - \alpha e^{-y_n})e^{-z_n} \\ y_{n+1} &= (\alpha x_n + \beta y_n)e^{-z_n} \\ z_{n+1} &= Ae^{-x_n} + Be^{-y_n}z_n \end{aligned} \tag{78}$$

where $\alpha < 1, \beta, A, B > 0, n = 0, 1, 2, \dots$ and $x_0, y_0, z_0 \geq 0$.

To discuss the dynamics of (78), we will first consider the case when $x = 0$. This restriction gives the system below in the yz plane.

$$\begin{aligned} y_{n+1} &= \beta y_n e^{-z_n} \\ z_{n+1} &= A + B e^{-y_n} z_n \end{aligned} \tag{79}$$

The fixed points of (79) satisfy :

$$\begin{aligned} y &= \beta y e^{-z} \\ z &= A + B e^{-y} z \end{aligned} \quad (80)$$

Thus (79) has two possible fixed points $(0, \frac{A}{1-B})$ when $B < 1$ and $(\ln(\frac{B \ln \beta}{\ln \beta - A}), \ln \beta)$ when $1 - B < \frac{A}{\ln \beta} < 1 < \beta$.

For convenience, set $(0, z_1) = (0, \frac{A}{1-B})$ and $(y_2, z_2) = (\ln(\frac{B \ln \beta}{\ln \beta - A}), \ln \beta)$

Theorem 18 *The hyperbolic fixed points of system (18) are given by the chart below,*

Case	$E_1(0, z_1)$	$E_2(y_2, z_2)$	Conditions	Global Behavior
1	Saddle	DNE	$B < 1, \ln \beta > \frac{A}{1-B}$	If $y_0 = 0$, $\lim_{n \rightarrow \infty} (y_n, z_n) = (0, \frac{A}{1-B})$. If $y_0 > 0$, $\lim_{n \rightarrow \infty} (y_n, z_n) = (\infty, A)$.
2	LAS	DNE	$B, \beta < 1$	$\lim_{n \rightarrow \infty} (y_n, z_n) = (0, \frac{A}{1-B})$
3	LAS	Saddle	$B < 1, y_2 < \frac{4 \ln \beta - 2A}{\ln \beta (\ln \beta - A)}$	$(0, \frac{A}{1-B})$ and (∞, A) have significant basins of attraction bounded by a monotonic curve.
4	LAS	Source	$B < 1, y_2 > \frac{4 \ln \beta - 2A}{\ln \beta (\ln \beta - A)}$	$(0, \frac{A}{1-B})$ and (∞, A) have significant basins of attraction bounded by a monotonic curve(s).
5	DNE	Saddle	$B > 1, y_2 < \frac{4 \ln \beta - 2A}{\ln \beta (\ln \beta - A)}$	$(0, \infty)$ and (∞, A) have significant basins of attraction bounded by a monotonic curve.
6	DNE	Source	$B > 1, y_2 > \frac{4 \ln \beta - 2A}{\ln \beta (\ln \beta - A)}$	$(0, \infty)$ and (∞, A) have significant basins of attraction bounded by a monotonic curve(s).
7	DNE	DNE	$B > 1, \beta < 1$	$\lim_{n \rightarrow \infty} (y_n, z_n) = (0, \infty)$

Proof. The characteristic polynomial given by the Jacobian of the map of (79) evaluated at E_1 is

$$\lambda^2 - (B + e^{-\frac{A}{1-B}})\lambda + B\beta e^{-\frac{A}{1-B}} = 0$$

Thus the two solutions are $\lambda_1 = B$ and $\lambda_2 = \beta e^{-\frac{A}{1-B}}$. Since $0 < B < 1$ is a condition for E_1 to exist, $|\lambda_1| < 1$ whenever E_1 exists, which are cases 1-4. So the stability character depends on $\beta e^{-\frac{A}{1-B}}$. So, $\beta e^{-\frac{A}{1-B}} > 1$ if and only if $\ln \beta > \frac{A}{1-B}$

thus E_1 is a saddle in this case 1. Similarly, $\beta e^{-\frac{A}{1-B}} < 1$ if and only if $\ln \beta < \frac{A}{1-B}$.

So E_1 is LAS in these cases 2-4.

The characteristic polynomial given by the Jacobian of the map of (79) evaluated at E_2 is

$$\lambda^2 - \left(2 - \frac{A}{\ln \beta}\right) \lambda - \left(\frac{A}{\ln \beta} + \ln \beta y_2 - A y_2 - 1\right) = 0$$

Next let $P = 2 - \frac{A}{\ln \beta}$ and $Q = \frac{A}{\ln \beta} + \ln \beta y_2 - A y_2 - 1$. Then $P > 1 - Q$ if and only if $0 > -\ln \beta y_2 + A y_2$ if and only if $\ln \beta > A$ which is always true. In addition, $P > Q - 1$ if and only if $4 - 2\frac{A}{\ln \beta} > y_2(\ln \beta - A)$ if and only if $y_2 < \frac{4\ln \beta - 2A}{\ln \beta(\ln \beta - A)}$. Thus by the Schur-Cohn conditions E_2 is a saddle point in cases 3 and 5 as $|P| > |1 - Q|$. Also, $P < Q - 1$ if and only if $4 - 2\frac{A}{\ln \beta} < y_2(\ln \beta - A)$ if and only if $y_2 > \frac{4\ln \beta - 2A}{\ln \beta(\ln \beta - A)}$. Additionally, $Q > 1$ if and only if $y_2(\ln \beta - A) > 2 - \frac{A}{\ln \beta}$ if and only if $y_2 > \frac{2\ln \beta - A}{\ln \beta(\ln \beta - A)}$, which is true since $y_2 > \frac{4\ln \beta - 2A}{\ln \beta(\ln \beta - A)}$. Thus by the Schur-Cohn conditions E_2 is a source in cases 4 and 6 as $|P| < |1 - Q|$ and $|Q| > 1$.

Now to discuss the global behavior of system (79). In case 1, if $y_0 = 0$ then (79) is reduced to $z_{n+1} = A + Bz_n$ whose solution is $\frac{A}{1-B}$ as $B < 1$. Now if $y_0 > 0$, then $\{y_n\} \rightarrow \infty$ as $n \rightarrow \infty$ since $\ln \beta > \frac{A}{1-B}$. Thus $\{z_n\} \rightarrow A$ as $n \rightarrow \infty$. Thus $\lim_{n \rightarrow \infty} (y_n, z_n) = (\infty, A)$. In case 2, since $\beta < 1$ and $e^{-z_n} < 1$ (as $z_n > 0$), $\{y_n\} \rightarrow 0$ as $n \rightarrow \infty$. Also since $B < 1$, $z_{n+1} = A + Bz_n$ whose solution is $\frac{A}{1-B}$. Thus $\lim_{n \rightarrow \infty} (y_n, z_n) = \left(0, \frac{A}{1-B}\right)$. In case 3, there is an interior fixed point which is a saddle point. Therefore by Theorem 1 in [20] there exists a curve through the fixed point which divides the basins of attraction of $\left(0, \frac{A}{1-B}\right)$ and (∞, A) . In case 4, there is an interior fixed point which is a source. Therefore by Theorem 2 in [20] there exists at most two curves whose endpoints are on the boundary which pass through the fixed point and bounds the basins of attraction of $\left(0, \frac{A}{1-B}\right)$ and (∞, A) . Case 5 is similar to case 3 only this curve separates the basins of attraction

of $(0, \infty)$ and (∞, A) . Case 6 is similar to case 4 except these curves will bound the basins of attraction of $(0, \infty)$ and (∞, A) . In case 7, since $\beta < 1$ and $e^{-z_n} < 1$ (as $z_n > 0$), $\{y_n\} \rightarrow 0$ as $n \rightarrow \infty$. Also since $B > 1$, $z_{n+1} = A + Bz_n$ whose solution is unbounded. Thus $\{z_n\} \rightarrow \infty$ as $n \rightarrow \infty$, so Thus $\lim_{n \rightarrow \infty} (y_n, z_n) = (0, \infty)$.
 \square

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